LECTURE NOTES
ON
ENGINEERING MECHANICS

B. Tech II Semester (MR-20)

Prepared By

Bolla Charitha
Assistant Professor

MECHANICAL ENGINEERING

MALLA REDDY ENGINEERING COLLEGE
(Autonomous)

ENGINEERING MECHANICS

II Semester: ME								
Course Code	Category	Hours / Week			Credits	Maximum Marks		
A0303	Core	L	T	P	C	CIA	SEE	Total
		3	0	0	3	30	70	100
Contact Classes: 48	Tutorial Classes: Nil	Practical Classes: Nil			Total Classes: 48			

COURSE OBJECTIVES:

The course should enable the students to:

I. Students should develop the ability to work comfortably with basic engineering mechanics concepts required for analyzing static structures.

II. Identify an appropriate structural system to studying a given problem and isolate it from its environment, model the problem using good free-body diagrams and accurate equilibrium equations.

III. Understand the meaning of centre of gravity (mass)/centroid and moment of Inertia using integration methods and method of moments

IV. To solve the problem of equilibrium by using the principle of work and energy, impulse momentum and vibrations for preparing the students for higher level courses such as Mechanics of Solids, Mechanics of Fluids, Mechanical Design and Structural Analysis etc...

MODULE-I	Introduction to Mechanics & System of Forces	Classes: 10

Introduction: Basic Concepts, Laws of Motion, Force - types, characteristics - Principle of transmissibility - Types of Forces - Concurrent and non-concurrent Forces - Composition of force – Resultant - Triangle, Polygon and Parallelogram Law of Forces - Moment of Force and its Application - Varignon‘s theorem, Couples - Free Body Diagrams, Types of Supports and their reactions, Internal and External Forces - Types of Equilibrium, Equations of Equilibrium, Conditions of Equilibrium - Lami‘s Theorem.

MODULE-II	Friction, Centroid and Center of Gravity	Classes: 10

Friction: Types of friction, Limiting friction, Laws of friction, static and dynamic friction, application of laws of friction. Motion of bodies - wedge, screw, screw jack. Centroid and Center of Gravity: Introduction, Centroids of Lines and Areas - simple figures - Centroid of composite figures. Pappus theorem - Centre of gravity of simple solids, composite solids - Centroids of volumes.

MODULE-III	Moment of Inertia	Classes: 10

A: Area Moment of Inertia: Definition - Moment of Inertia of plane areas, Polar Moment of Inertia, Transfer Theorem, Moments of Inertia of Composite Figures. B: Mass Moment of Inertia: Introduction-moment of inertia of masses - Radius of gyration-Transfer formula for mass moment of inertia- by integration - Moment of Inertia of composite bodies.

MODULE-IV	Kinematics & Kinetics	Classes: 09

Kinematics: Rectilinear motion - Motion of Rigid Body under uniform and variable accelerations - motion under gravity- curvilinear motion – Projectiles - rotary motion. Kinetics: Analysis as a Particle and Analysis as a Rigid Body in Translation - D’Alemberts Principle - Connected bodies- Kinetics of rotating bodies.

MODULE-V	Work, Power, Energy & Mechanical Vibrations	Classes: 09
Work, Power and Energy: Introduction, work-energy equation - motion of connected bodies - work done by a spring - general plane motion. Mechanical Vibrations: Definitions, concepts - simple harmonic motion - free vibrations - Simple and compound pendulums.		

Text Books:

1. S. Timoshenko, D.H. Young, J.V. Rao and Sukumar Pati, "Engineering Mechanics", Tata McGraw-Hill Education, 5th Edition, 2013.
2. K.Vijaya Kumar Reddy, J. Suresh Kumar, "Engineering Mechanics", B S Publications, 3 rd Edition, 2013.

Reference Books:

1. Beer, F.P and Johnston Jr. E.R. "Vector Mechanics for Engineers", Tata McGraw-Hill Education 10th Edition (India) Pvt Ltd.. 2013.
2. Fedinand. L. Singer, "Engineering Mechanics", Harper & Row Publishers, 3rd Edition, 1975.
3. S.Bhavikatti,"ATextBookofEngineeringMechanics",NewAgeInternational,1st Edition,2012
4. R.S. Khurmi, "A Text Book of Engineering Mechanics", S.Chand Publications, 21st Edition, 2007.
5. K L Kumar, "Engineering Mechanics", Tata McGraw Hill Education, 4th Edition, 2011

Web References:

1. http://nptel.ac.in/courses/112103109/
2. http://nptel.ac.in/courses/112106180/
3. http://nptel.ac.in/courses/115104094/

E-Text Books:

1. http://www.mathalino.com/reviewer/engineering-mechanics/equilibrium-force-system
2. http://ascelibrary.org/journal/jenmdt
3. https://tll.mit.edu/sites/default/files/SUTDVideoThumb/freebodydiagrams.pdf

MODULE I

Introduction to Mechanics & System of Forces

Mechanics

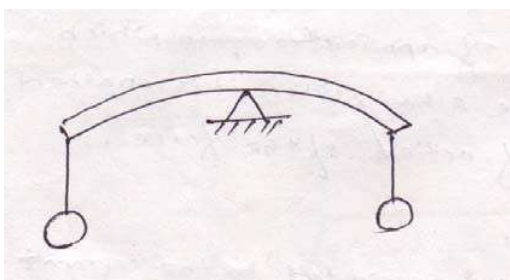
It is defined as that branch of science, which describes and predicts the conditions of rest or motion of bodies under the action of forces. Engineering mechanics applies the principle of mechanics to design, taking into account the effects of forces.

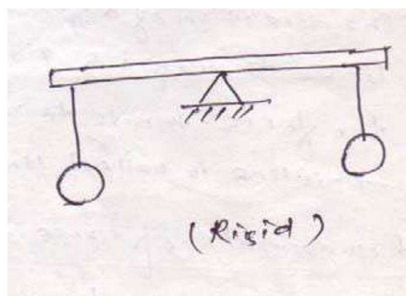
Statics

Statics deal with the condition of equilibrium of bodies acted upon by forces.

Rigid body

A rigid body is defined as a definite quantity of matter, the parts of which are fixed in position relative to each other. Physical bodies are never absolutely but deform slightly under the action of loads. If the deformation is negligible as compared to its size, the body is termed as rigid.



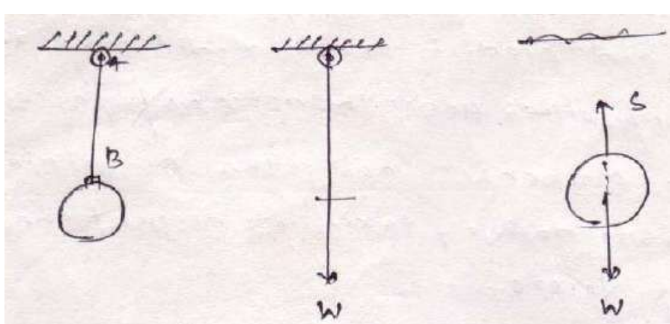


Force

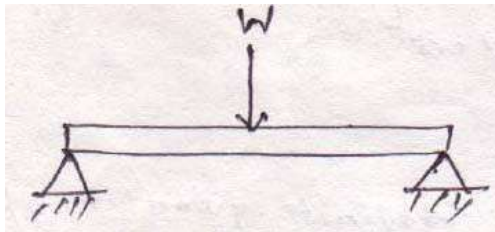
Force may be defined as any action that tends to change the state of rest or motion of a body to which it is applied.

The three quantities required to completely define force are called its specification or characteristics. So the characteristics of a force are:

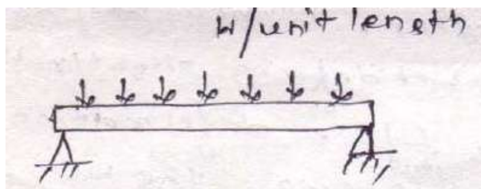
1. Magnitude
2. Point ofapplication
3. Direction ofapplication



Concentrated force/point load



Distributed force

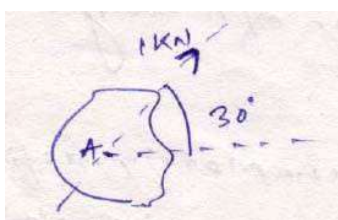


Line of action of force

The direction of a force is the direction, along a straight line through its point of application in which the force tends to move a body when it is applied. This line is called line of action of force.

Representation of force

Graphically a force may be represented by the segment of a straight line.

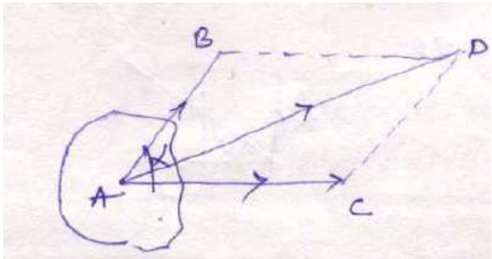


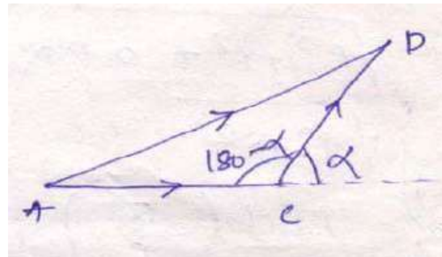
Composition of two forces

The reduction of a given system of forces to the simplest system that will be its equivalent is called the problem of composition of forces.

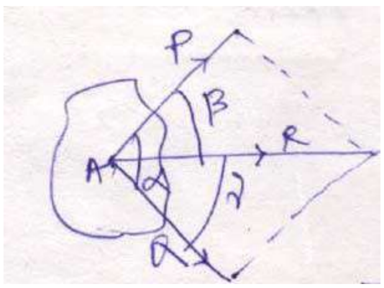
Parallelogram law

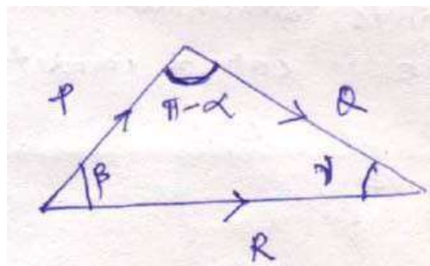
If two forces represented by vectors AB and AC acting under an angle α are applied to a body at point A. Their action is equivalent to the action of one force, represented by vector AD, obtained as the diagonal of the parallelogram constructed on the vectors AB and AC directed as shown in thefigure.





Force AD is called the resultant of AB and AC and the forces are called its components.





$$R = \sqrt{\left(P^2+Q^2+2PQ\times Cos\alpha\right)}$$

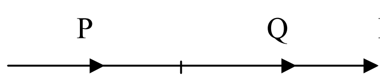
Now applying triangle law

$$\frac{P}{Sin\gamma} = \frac{Q}{Sin\beta} = \frac{R}{Sin(\pi-\alpha)}$$

Special cases

Case-I: If α = 0°

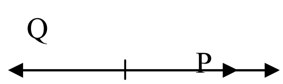
$$R = \sqrt{\left(P^2+Q^2+2PQ\times Cos0^\circ\right)} = \sqrt{(P+Q)^2} = (P+Q)$$



R = P+Q

Case- II: If α = 180°

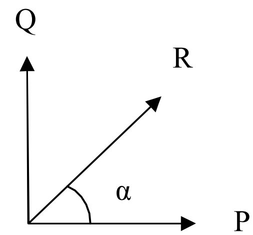
$$R = \sqrt{\left(P^2+Q^2+2PQ\times Cos180^\circ\right)} = \sqrt{(P^2+Q^2-2PQ)} = \sqrt{(P-Q)^2} = (P-Q)$$



Case-III: If α = 90˚

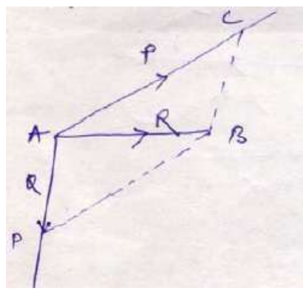
$$R = \sqrt{\left(P^2 + Q^2 + 2PQ \times Cos90^\circ\right)} = \sqrt{P^2 + Q^2}$$

$\alpha = \tan^{-1}(Q/P)$



Resolution of a force

The replacement of a single force by a several components which will be equivalent in action to the given force is called resolution of a force.



Action and reaction

Often bodies in equilibrium are constrained to investigate the conditions.

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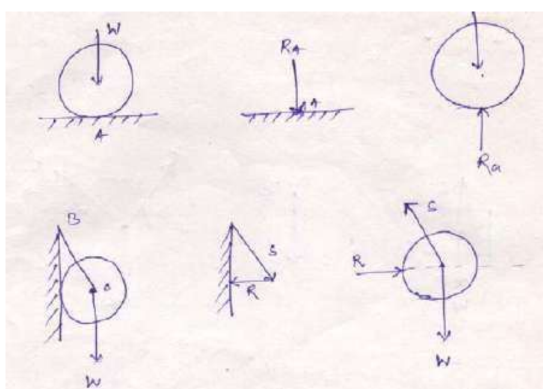
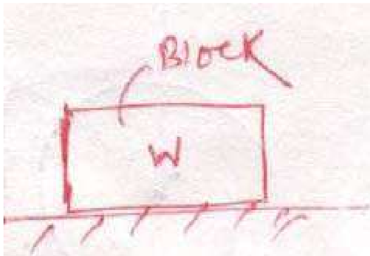
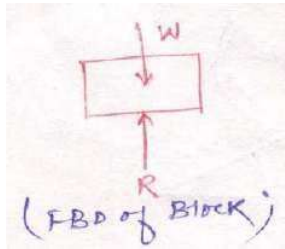


Free body diagram

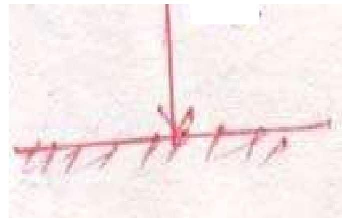
Free body diagram is necessary to investigate the condition of equilibrium of a body or system. While drawing the free body diagram all the supports of the body are removed and replaced with the reaction forces acting on it.

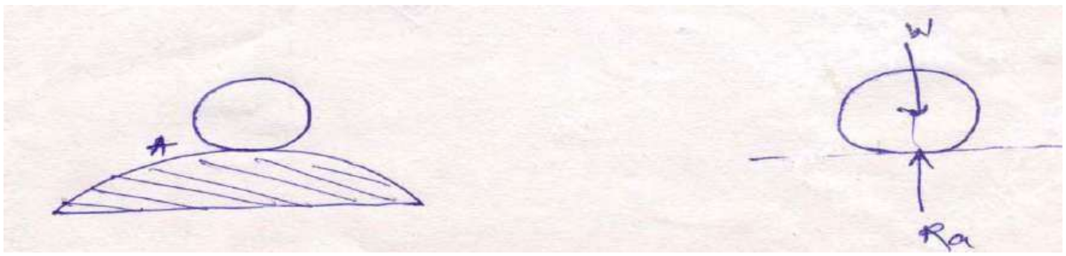
1. Draw the free body diagrams of the followingfigures.



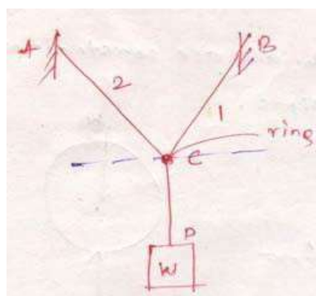


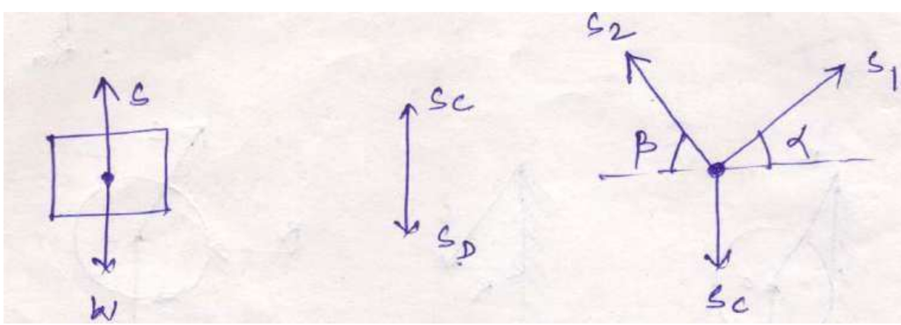
R



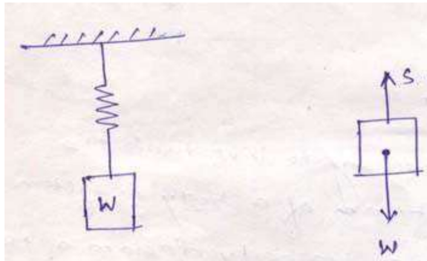


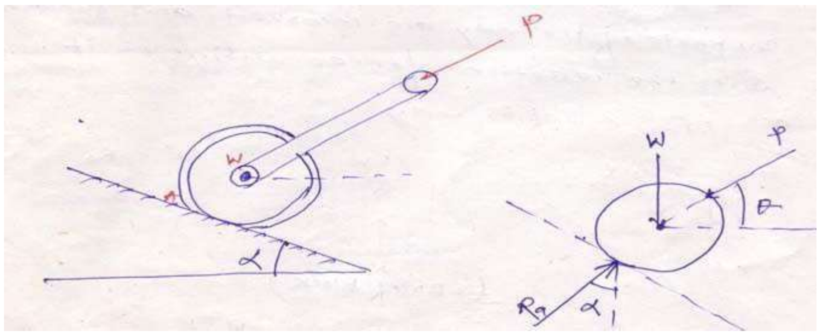
2. Draw the free body diagram of the body, the string CD and thering.





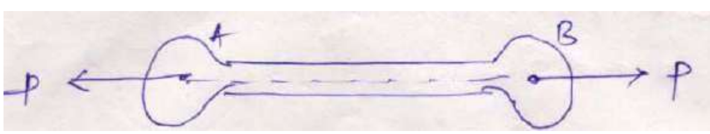
3. Draw the free body diagram of the following figures.



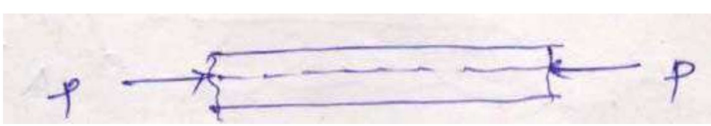


Equilibrium of colinear forces:

Equilibrium law: Two forces can be in equilibrium only if they are equal in magnitude, opposite in direction and collinear in action.



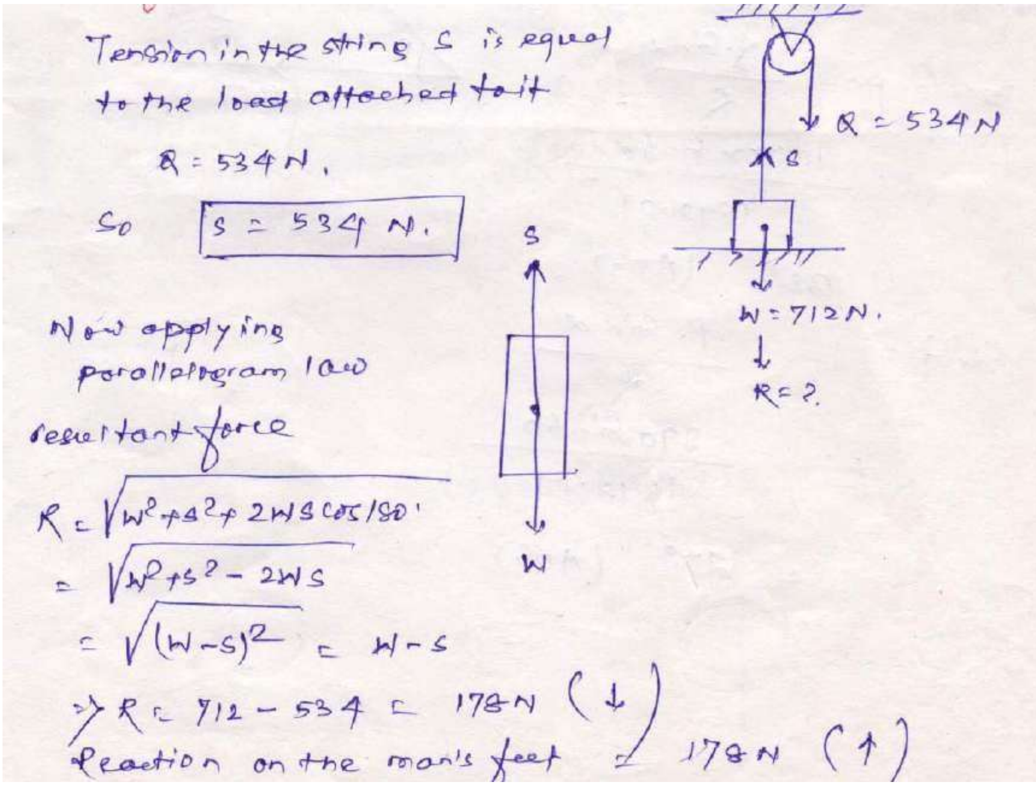
(tension)



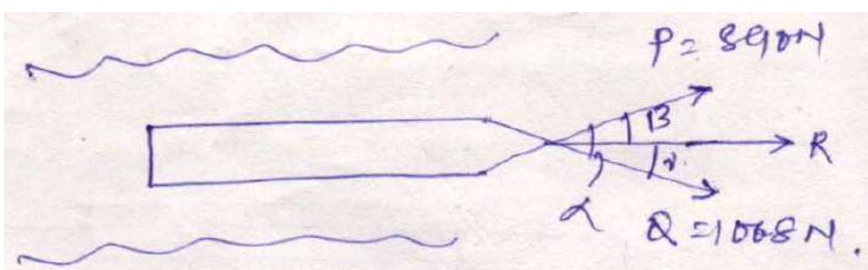
(compression)

Superposition and transmissibility

Problem 1: A man of weight W = 712 N holds one end of a rope that passes over a pulley vertically above his head and to the other end of which is attached a weight Q = 534 N. Find the force with which the man's feet press against the floor.

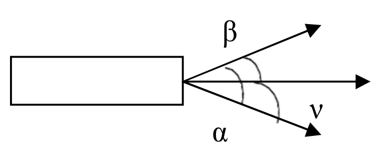


Problem 2: A boat is moved uniformly along a canal by two horses pulling with forces P = 890 N and Q = 1068 N acting under an angle α = 60˚. Determine the magnitude of the resultant pull on the boat and the angles β and ν.



P = 890 N, α = 60˚
Q = 1068 N

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$
$$= \sqrt{890^2 + 1068^2 + 2\times 890\times 1068\times 0.5}$$
$$= 1698.01\,N$$

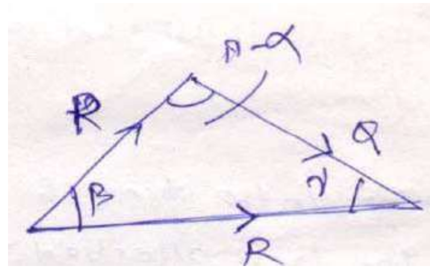


$$\frac{Q}{\sin\beta}=\frac{P}{\sin v}=\frac{R}{\sin(\pi-\alpha)}$$

$$\sin\beta=\frac{Q\sin\alpha}{R}$$

$$=\frac{1068\times\sin60^\circ}{1698.01}$$

$$=33^\circ$$



$$\sin v=\frac{P\sin\alpha}{R}$$

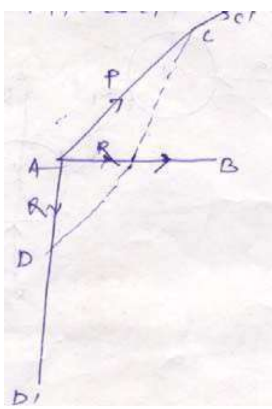
$$=\frac{890\times\sin60^\circ}{1698.01}$$

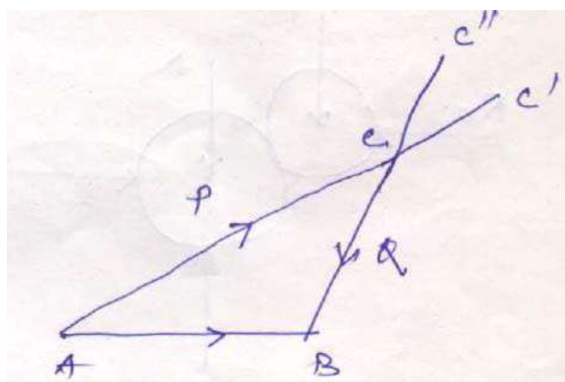
$$=27^\circ$$

<u>Resolution of a force</u>

Replacement of a single force by several components which will be equivalent in action to the given force is called the problem of resolution of aforce.

By using parallelogram law, a single force R can be resolved into two components P and Q intersecting at a point on its line of action.





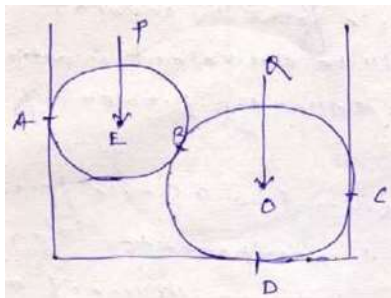
<u>Equilibrium of collinear forces:</u>

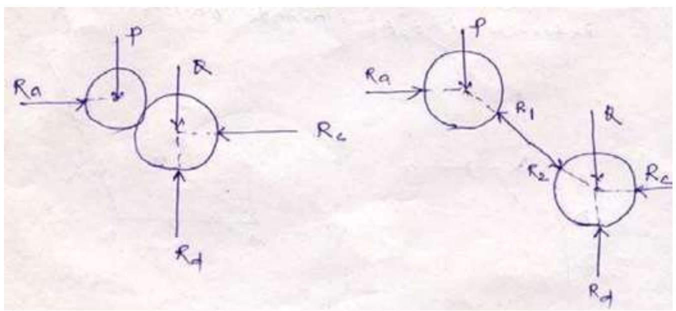
Equilibrium law: Two forces can be in equilibrium only if they are equal in magnitude, opposite in direction and collinear in action.

Law of superposition

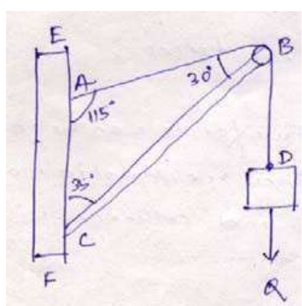
The action of a given system of forces on a rigid body will no way be changed if we add to or subtract from them another system of forces in equllibrium.

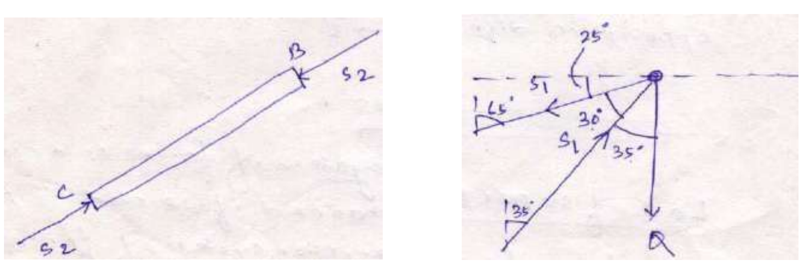
Problem 3: Two spheres of weight P and Q rest inside a hollow cylinder which is resting on a horizontal force. Draw the free body diagram of both the spheres, together and separately.



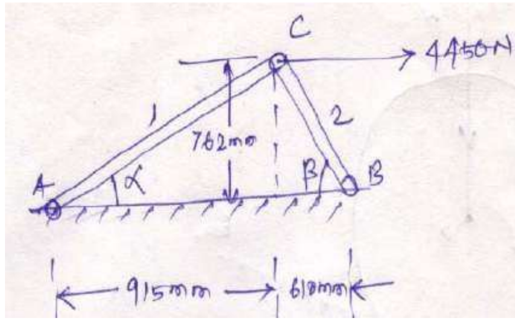


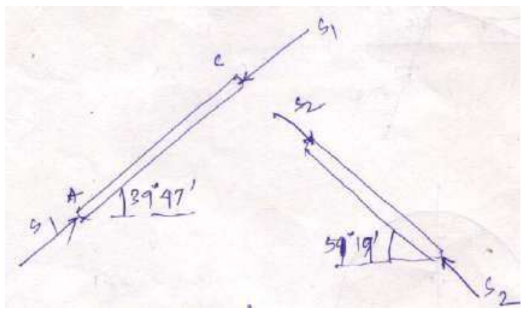
Problem 4: Draw the free body diagram of the figure shown below.



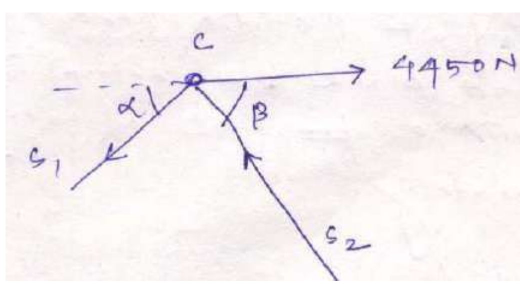


Problem 5: Determine the angles α and β shown in the figure.

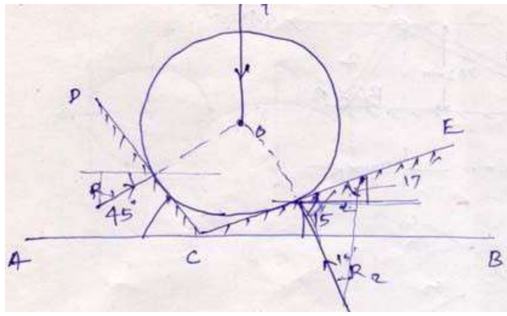


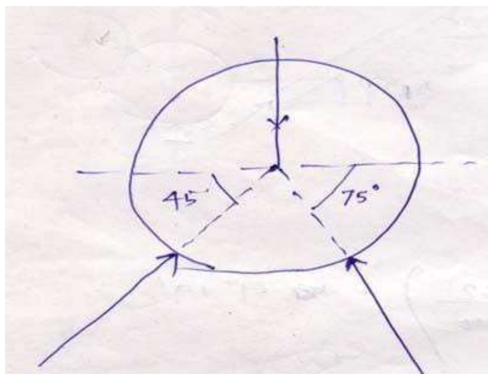


$$\alpha = \tan^{-1}\left(\frac{762}{915}\right)$$
$$= 39^\circ 47'$$
$$\beta = \tan^{-1}\left(\frac{762}{610}\right)$$
$$= 51^\circ 19'$$

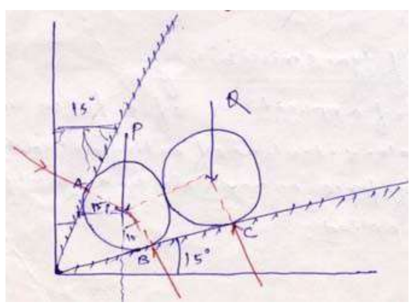


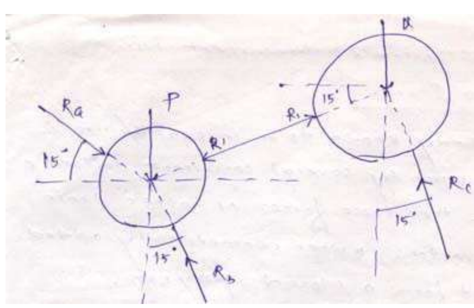
Problem 6: Find the reactions R_1 and R_2.



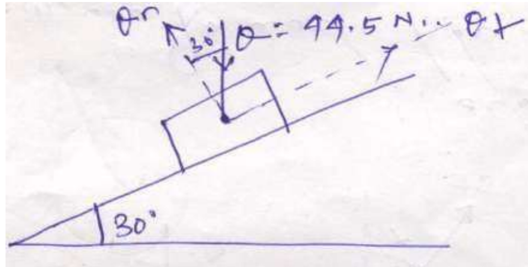


Problem 7: Two rollers of weight P and Q are supported by an inclined plane and vertical walls as shown in the figure. Draw the free body diagram of both the rollers separately.

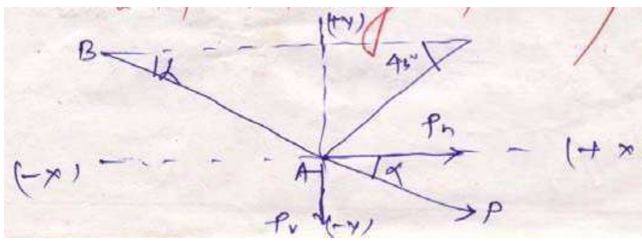




Problem 8: Find θ_n and θ_t in the following figure.



Problem 9: For the particular position shown in the figure, the connecting rod BA of an engine exert a force of P = 2225 N on the crank pin at A. Resolve this force into two rectangularcomponents$P_h$and$P_v$horizontallyandverticallyrespectivelyatA.

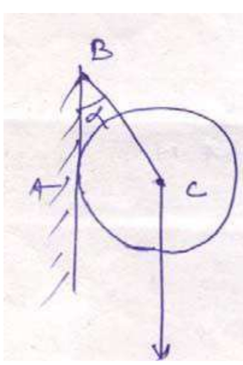


P_h = 2081.4 N
P_v = 786.5 N

Equilibrium of concurrent forces in a plane

- If a body known to be in equilibrium is acted upon by several concurrent, coplanar forces, then these forces or rather their free vectors, when geometrically added must form a closedpolygon.

- This system represents the condition of equilibrium for any system of concurrent forces in aplane.



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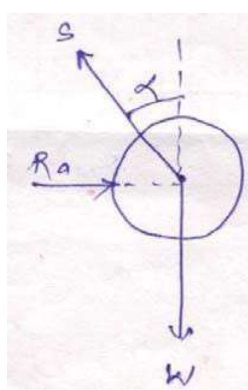
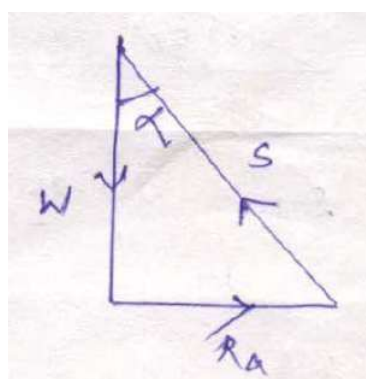
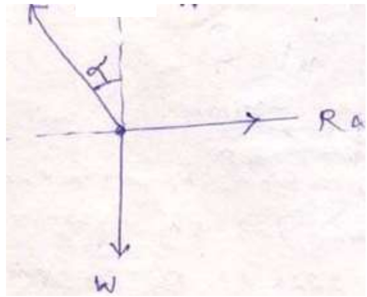




$R_a = w\tan\alpha$

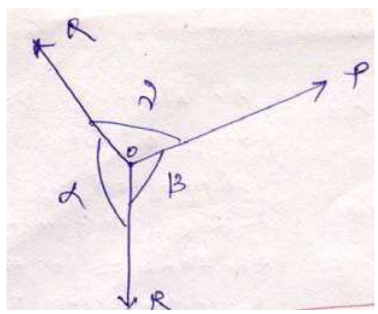
$S = w\sec\alpha$

S

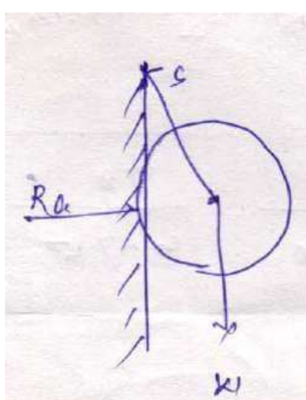


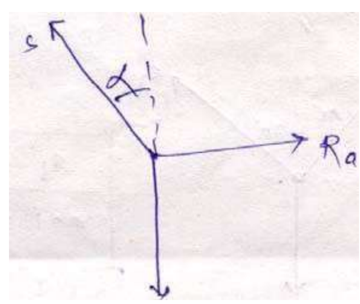
Lami's theorem

If three concurrent forces are acting on a body kept in an equllibrium, then each force is proportional to the sine of angle between the other two forces and the constant of proportionality issame.



$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\upsilon}$$

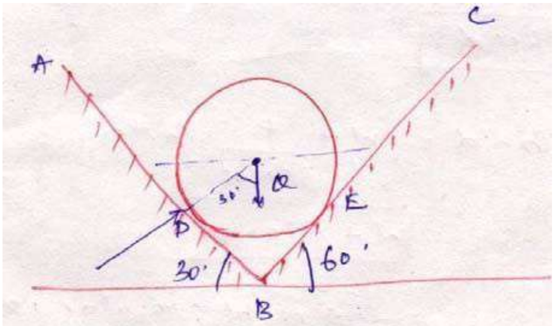


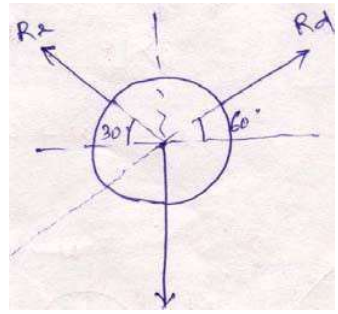


W

$$\frac{S}{\sin 90} = \frac{R_a}{\sin(180-\alpha)} = \frac{W}{\sin(90+\alpha)}$$

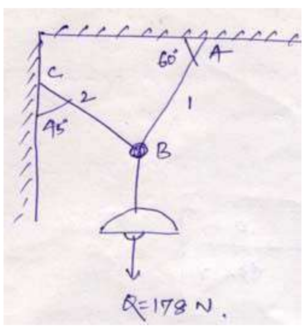
Problem: A ball of weight Q = 53.4N rest in a right angled trough as shown in figure. Determine the forces exerted on the sides of the trough at D and E if all the surfaces are perfectlysmooth.

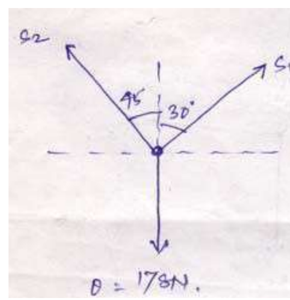




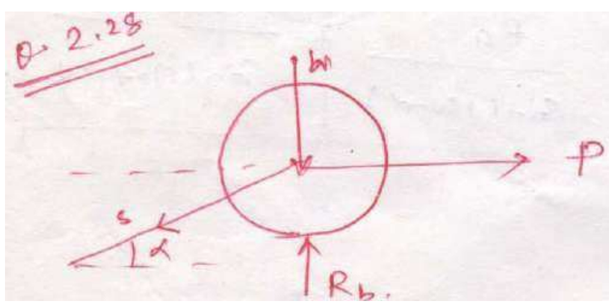
W

Problem: An electric light fixture of weight Q = 178 N is supported as shown in figure. Determine the tensile forces S_1 and S_2 in the wires BA and BC, if their angles of inclination aregiven.





$$\frac{S_1}{\sin 135} = \frac{S_2}{\sin 150} = \frac{178}{\sin 75}$$

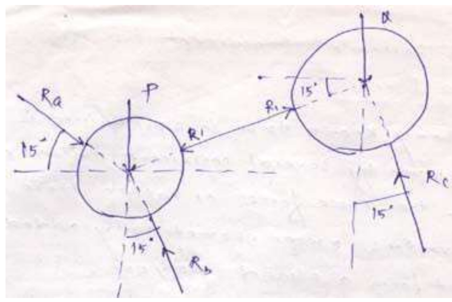


$S_1 \cos\alpha = P$

$\boxed{S = P\sec\alpha}$

$$R_b = W + S \sin \alpha$$
$$= W + \frac{P}{\cos\alpha} \times \sin \alpha$$
$$= W + P \tan \alpha$$

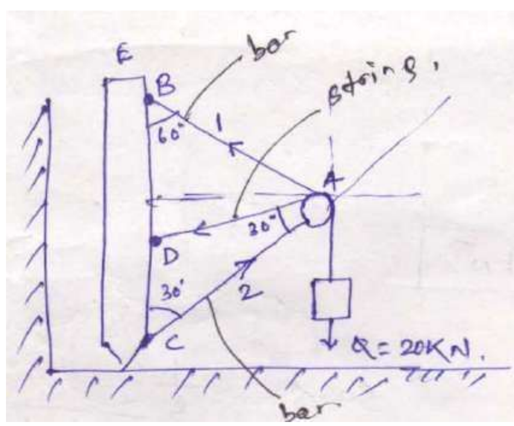
Problem: A right circular roller of weight W rests on a smooth horizontal plane and is held in position by an inclined bar AC. Find the tensions in the bar AC and vertical reaction R_b if there is also a horizontal force P is active.

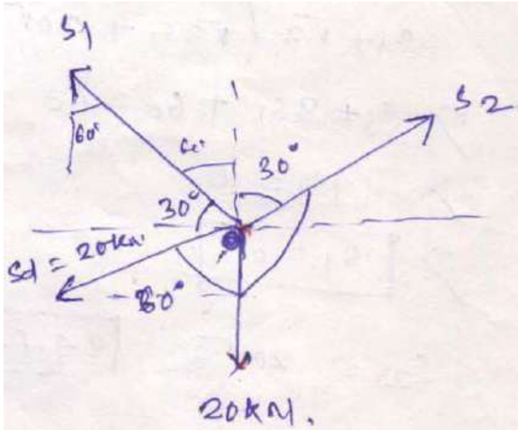


<u>**Theory of transmissiibility of a force:**</u>

The point of application of a force may be transmitted along its line of action without changing the effect of force on any rigid body to which it may be applied.

Problem:





$$\sum X = 0$$

$$S_1 \cos 30 + 20 \sin 60 = S_2 \sin 30$$

$$\frac{\sqrt{3}}{2} S_1 + 20 \frac{\sqrt{3}}{2} = \frac{S_2}{2}$$

$$\frac{S_2}{2} = \frac{\sqrt{3}}{2} S_1 + 10\sqrt{3}$$

$$S_2 = \sqrt{3} S_1 + 20\sqrt{3} \qquad (1)$$

$$\sum Y = 0$$

$$S_1 \sin 30 + S_2 \cos 30 = S_d \cos 60 + 20$$

$$\frac{S_1}{2} + S_2 \frac{\sqrt{3}}{2} = \frac{20}{2} + 20$$

$$\frac{S_1}{2} + \frac{\sqrt{3}}{2} S_2 = 30$$

$$S_1 + \sqrt{3} S_2 = 60 \qquad (2)$$

Substituting the value of S_2 in Eq.2, we get

$$S_1 + \sqrt{3}\left(\sqrt{3} S_1 + 20\sqrt{3}\right) = 60$$

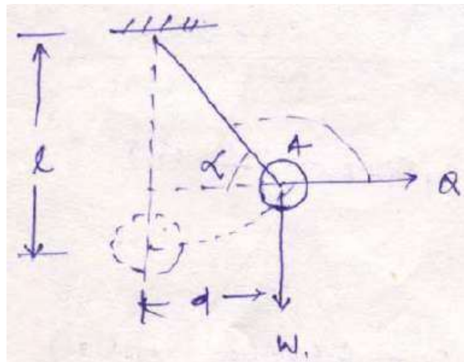
$$S_1 + 3S_1 + 60 = 60$$

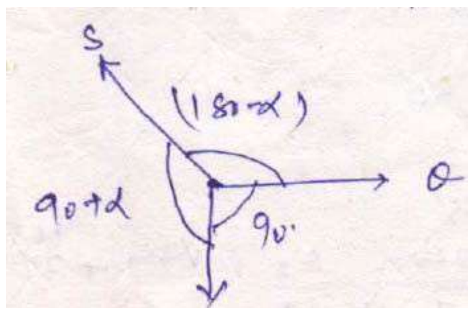
$$4S_1 = 0$$

$$S_1 = 0KN$$

$$S_2 = 20\sqrt{3} = 34.64KN$$

Problem: A ball of weight W is suspended from a string of length l and is pulled by a horizontal force Q. The weight is displaced by a distance d from the vertical position as shown in Figure. Determine the angle α, forces Q and tension in the string S in the displacedposition.





W

$$\cos\alpha = \frac{d}{l}$$

$$\alpha = \cos^{-1}\left(\frac{d}{l}\right)$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\Rightarrow \sin\alpha = \sqrt{(1-\cos^2\alpha)}$$

$$= \sqrt{1-\frac{d^2}{l^2}}$$

$$= \frac{1}{l}\sqrt{l^2-d^2}$$

Applying Lami's theorem,

$$\frac{S}{\sin 90} = \frac{Q}{\sin(90+\alpha)} = \frac{W}{\sin(180-\alpha)}$$

$$\frac{Q}{\sin(90+\alpha)} = \frac{W}{\sin(180-\alpha)}$$

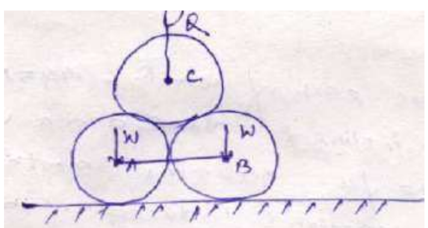
$$\Rightarrow Q = \frac{W\cos\alpha}{\sin\alpha} = \frac{W\left(\frac{d}{l}\right)}{\frac{1}{l}\sqrt{l^2-d^2}}$$

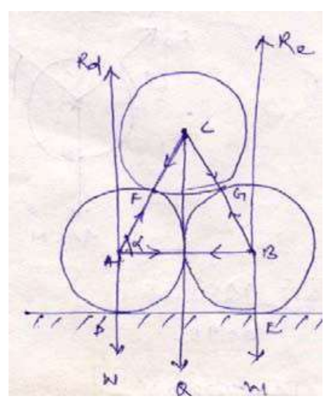
$$\Rightarrow Q = \frac{Wd}{\sqrt{l^2-d^2}}$$

$$S = \frac{W}{\sin\alpha} = \frac{W}{\frac{1}{l}\sqrt{l^2-d^2}}$$

$$= \frac{Wl}{\sqrt{l^2-d^2}}$$

<u>Problem:</u> Two smooth circular cylinders each of weight W = 445 N and radius r = 152 mm are connected at their centres by a string AB of length l = 406 mm and rest upon a horizontal plane, supporting above them a third cylinder of weight Q = 890 N and radius r = 152 mm. Find the forces in the string and the pressures produced on the floor at the point of contact.

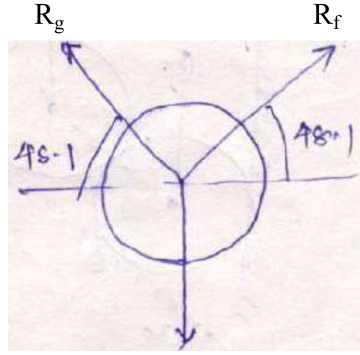




$$\cos\alpha = \frac{203}{304}$$

$$\Rightarrow \alpha = 48.1^{\circ}$$

R_g R_f



Q

$$\frac{R_g}{\sin 138.1} = \frac{R_e}{\sin 138.1} = \frac{Q}{83.8}$$

$$\Rightarrow R_g = R_e = 597.86N$$

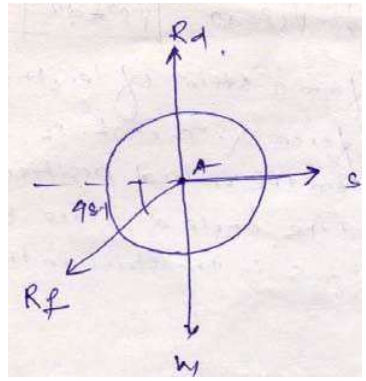
Resolving horizontally

$\sum X = 0$

$S = R_f \cos 48.1$

$= 597.86 \cos 48.1$

$= 399.27N$



Resolving vertically

$\sum Y = 0$

$R_d = W + R_f \sin 48.1$

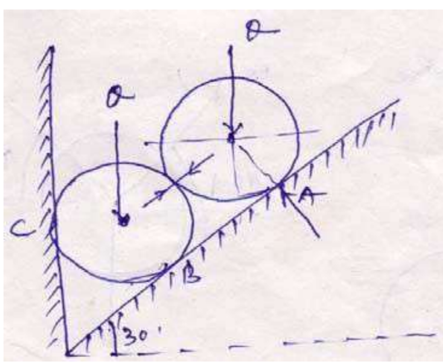
$= 445 + 597.86 \sin 48.1$

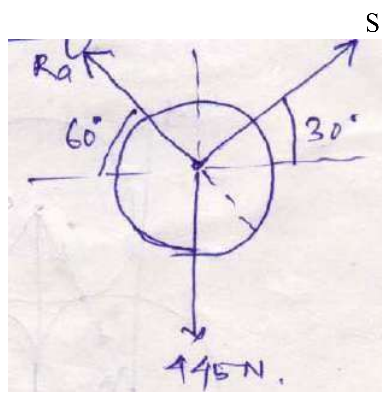
$= 890N$

$R_e = 890N$

$S = 399.27N$

Problem: Two identical rollers each of weight Q = 445 N are supported by an inclined plane and a vertical wall as shown in the figure. Assuming smooth surfaces, find the reactions induced at the points of support A, B and C.





$$\frac{R_a}{\sin 120} = \frac{S}{\sin 150} = \frac{445}{\sin 90}$$

$\Rightarrow R_a = 385.38N$

$\Rightarrow S = 222.5N$

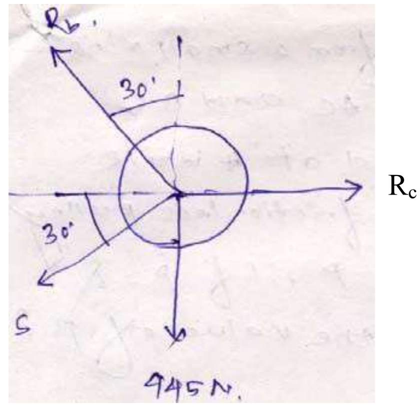
Resolving vertically

$\sum Y = 0$

$R_b \cos 60 = 445 + S \sin 30$

$\Rightarrow R_b \dfrac{\sqrt{3}}{2} = 445 + \dfrac{222.5}{2}$

$\Rightarrow R_b = 642.302N$



Resolving horizontally

$\sum X = 0$

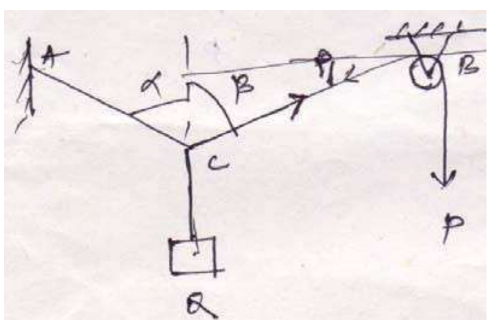
$R_c = R_b \sin 30 + S \cos 30$

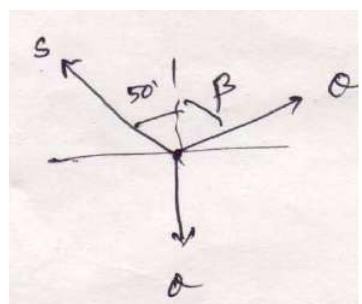
$\Rightarrow 642.302 \sin 30 + 222.5 \cos 30$

$\Rightarrow R_c = 513.84N$

Problem:

A weight Q is suspended from a small ring C supported by two cords AC and BC. The cord AC is fastened at A while cord BC passes over a frictionless pulley at B and carries a weight P. If P = Q and $\alpha = 50^\circ$, find the value of β.





Resolving horizontally

$\sum X = 0$

$S \sin 50 = Q \sin \beta$ (1)

Resolving vertically

$\sum Y = 0$

$S \cos 50 + Q \sin \beta = Q$

$\Rightarrow S \cos 50 = Q(1 - \cos \beta)$

Putting the value of S from Eq. 1, weget

$S\cos 50 + Q\sin\beta = Q$

$\Rightarrow S\cos 50 = Q(1-\cos\beta)$

$\Rightarrow Q\dfrac{\sin\beta}{\sin 50}\cos 50 = Q(1-\cos\beta)$

$\Rightarrow \cot 50 = \dfrac{1-\cos\beta}{\sin\beta}$

$\Rightarrow 0.839\sin\beta = 1-\cos\beta$

Squaring both sides,

$0.703\sin^2\beta = 1+\cos^2\beta - 2\cos\beta$

$0.703(1-\cos^2\beta) = 1+\cos^2\beta - 2\cos\beta$

$0.703 - 0.703\cos^2\beta = 1+\cos^2\beta - 2\cos\beta$

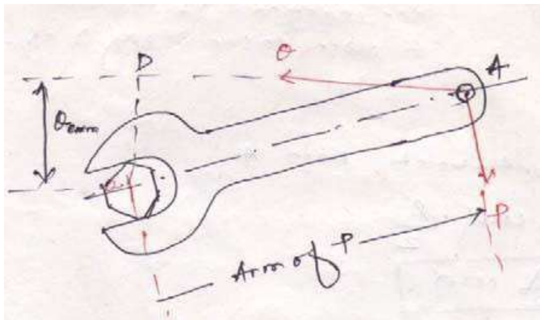
$\Rightarrow 1.703\cos^2\beta - 2\cos\beta + 0.297 = 0$

$\Rightarrow \cos^2\beta - 1.174\cos\beta + 0.297 = 0$

$\Rightarrow \beta = 63.13^{\circ}$

Method of moments

Moment of a force with respect to a point:



- Considering wrench subjected to two forces P and Q of equal magnitude. It is evident that force P will be more effective compared to Q, though they are of equalmagnitude.
- The effectiveness of the force as regards it is the tendency to produce rotation of a body about a fixed point is called the moment of the force with respect to that point.
- Moment = Magnitude of the force × Perpendicular distance of the line of action offorce.
- Point O is called moment centre and the perpendicular distance (i.e. OD) is called momentarm.
- Unit isN.m

Theorem of Varignon:

The moment of the resultant of two concurrent forces with respect to a centre in their plane is equal to the alzebric sum of the moments of the components with respect to some centre.

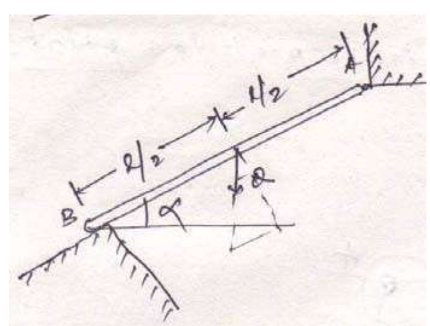
Problem 1:

A prismatic clear of AB of length l is hinged at A and supported at B. Neglecting friction, determine the reaction R_b produced at B owing to the weight Q of the bar.

Taking moment about point A,

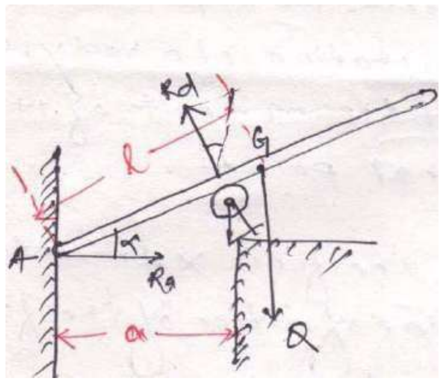
$$R_b \times l = Q\cos\alpha.\frac{l}{2}$$

$$\Rightarrow R_b = \frac{Q\cos\alpha}{2}$$



Problem 2:

A bar AB of weight Q and length 2l rests on a very small friction less roller at D and against a smooth vertical wall at A. Find the angle α that the bar must make with the horizontal in equilibrium.



Resolving vertically,

$$R_d \cos\alpha = Q$$

Now taking moment about A,

$$\frac{R_d . a}{\cos\alpha} - Q.l\cos\alpha = 0$$

$$\Rightarrow \frac{Q.a}{\cos^2\alpha} - Q.l\cos\alpha = 0$$

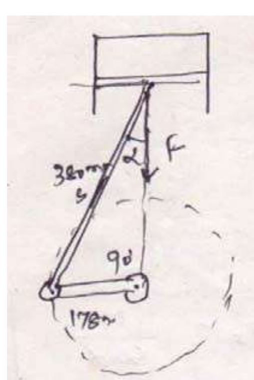
$$\Rightarrow Q.a - Q.l\cos^3\alpha = 0$$

$$\Rightarrow \cos^3\alpha = \frac{Q.a}{Q.l}$$

$$\Rightarrow \alpha = \cos^{-1}\sqrt[3]{\frac{a}{l}}$$

Problem 3:

If the piston of the engine has a diameter of 101.6 mm and the gas pressure in the cylinder is 0.69 MPa. Calculate the turning moment M exerted on the crankshaft for the particular configuration.



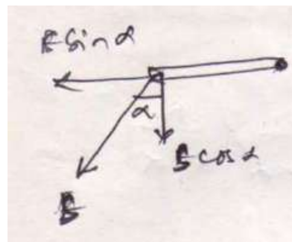
Area of cylinder

$$A = \frac{\pi}{4}(0.1016)^2 = 8.107\times10^{-3}\ m^2$$

Force exerted on connectingrod,

F = Pressure × Area
 $= 0.69\times10^6 \times 8.107\times10^{-3}$
 $= 5593.83$ N

Now $\alpha = \sin^{-1}\left(\frac{178}{380}\right) = 27.93^\circ$



$S\cos\alpha = F$

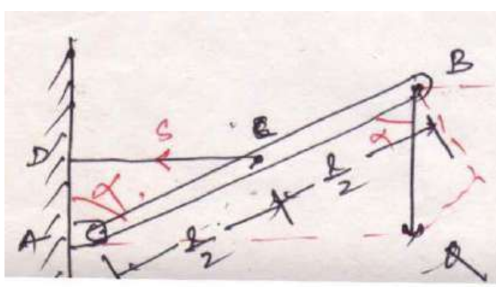
$$\Rightarrow S = \frac{F}{\cos\alpha} = 6331.29N$$

Now moment entered on crankshaft,

$S\cos\alpha \times 0.178 = 995.7N = 1KN$

Problem 4:

A rigid bar AB is supported in a vertical plane and carrying a load Qat its free end. Neglecting the weight of bar, find the magnitude of tensile force S in the horizontal string CD.



Taking moment about A,

$\sum M_A = 0$

$S.\frac{l}{2}\cos\alpha = Q.l\sin\alpha$

$$\Rightarrow S = \frac{Q.l\sin\alpha}{\frac{l}{2}\cos\alpha}$$

$\Rightarrow S = 2Q.\tan\alpha$

MODULE II

Friction, Centroid and Center of Gravity

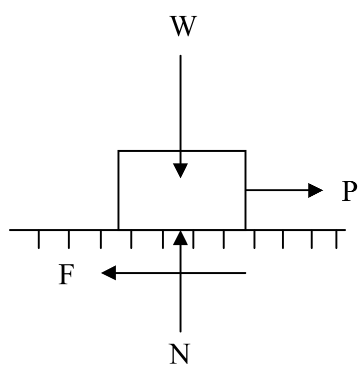
Friction

- The force which opposes the movement or the tendency of movement is called **Frictional force or simply friction**. It is due to the resistance to motion offered by minutely projecting particles at the contact surfaces. However, there is a limit beyond which the magnitude of this force cannotincrease.
- If the applied force is more than this limit, there will be movement of one body over the other. This limiting value of frictional force when the motion is impending, it is known as **LimitingFriction**.
- When the applied force is less than the limiting friction, the body remains at rest and such frictional force is called **Static Friction**, which will be having any value between zero and the limitingfriction.
- If the value of applied force exceeds the limiting friction, the body starts moving over the other body and the frictional resistance experienced by the body while moving is known as **Dynamic Friction**. Dynamic friction is less than limitingfriction.
- Dynamic friction is classified into following twotypes:

a) Slidingfriction

b) Rolling friction

- Sliding friction is the friction experienced by a body when it slides over the other body.
- Rolling friction is the friction experienced by a body when it rolls over a surface.
- It is experimentally found that the magnitude of limiting friction bears a constant ratio to the normal reaction between two surfaces and this ratio is called **Coefficientof Friction**.



Coefficient of friction = $\frac{F}{N}$

where F is limiting friction and N is normal reaction between the contact surfaces.

Coefficient of friction is denoted by μ.

Thus, $\mu = \frac{F}{N}$

Laws of friction

1. The force of friction always acts in a direction opposite to that in which body tends tomove.
2. Till the limiting value is reached, the magnitude of friction is exactly equal to the force which tends to move thebody.
3. The magnitude of the limiting friction bears a constant ratio to the normal reaction between the two surfaces of contact and this ratio is called coefficient offriction.
4. The force of friction depends upon the roughness/smoothness of thesurfaces.
5. The force of friction is independent of the area of contact between the two surfaces.
6. After the body starts moving, the dynamic friction comes into play, the magnitude of which is less than that of limiting friction and it bears a constant ratio with normal force. This ratio is called **coefficient of dynamicfriction**.

Angle of friction

Consider the block shown in figure resting on a horizontal surface and subjected to horizontal pull P. Let F be the frictional force developed and N the normal reaction. Thus, at contact surface the reactions are F and N. They can be graphically combined to get the reaction R which acts at angle θ to normal reaction. This angle θ called the angle of friction is givenby

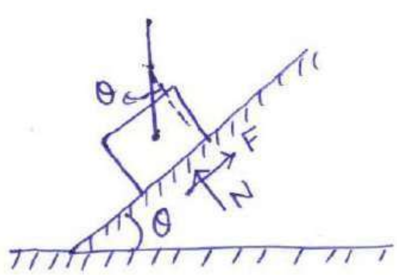
$$\tan\theta = \frac{F}{N}$$

As P increases, F increases and hence θ also increases. θ can reach the maximum value α when F reaches limiting value. At this stage,

$$\tan\alpha = \frac{F}{N} = \mu$$

This value of α is called Angle of Limiting Friction. Hence, the angle of limiting friction may be defined as the angle between the resultant reaction and the normal to the plane on which the motion of the body is impending.

Angle of repose



Consider the block of weight W resting on an inclined plane which makes an angle θ with the horizontal. When θ is small, the block will rest on the plane. If θ is gradually increased, a stage is reached at which the block start sliding down the plane. The angle θ for which the motion is impending, is called the angle of repose. Thus, the maximum inclination of the plane on which a body, free from external forces, can repose is called **Angle of Repose**.

Resolving vertically,
N = W. cos θ

Resolving horizontally,
F = W. sin θ

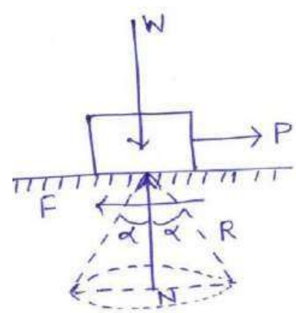
Thus, $\tan \theta = \frac{F}{N}$

If ϕ is the value of θ when the motion is impending, the frictional force will be limiting friction and hence,

$$\tan\varphi = \frac{F}{N}$$
$$= \mu = \tan\alpha$$
$$\Rightarrow \varphi = \alpha$$

Thus, the value of angle of repose is same as the value of limiting angle of repose.

Cone of friction

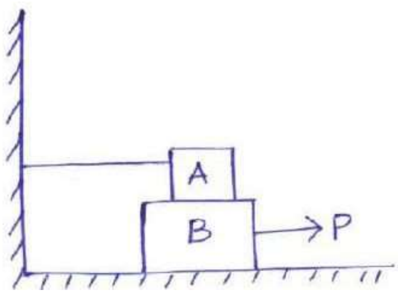


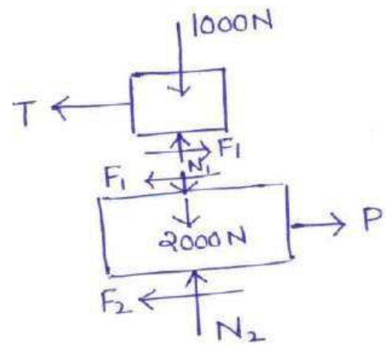
- When a body is having impending motion in the direction of force P, the frictional force will be limiting friction and the resultant reaction R will make limiting angle α with thenormal.
- If the body is having impending motion in some other direction, the resultant reaction makes limiting frictional angle α with the normal to that direction. Thus, when the direction of force P is gradually changed through 360˚, the resultant R generates a right circular cone with semi-central angle equal toα.

Problem 1: Block A weighing 1000N rests over block B which weighs 2000N as shown in figure. Block A is tied to wall with a horizontal string. If the coefficient of friction between blocks A and B is 0.25 and between B and floor is 1/3, what should be the value of P to move the block (B), if

(a) P ishorizontal.
(b) P acts at 30˚ upwards tohorizontal.

Solution: (a)





Considering block A,

$$\sum V = 0$$
$$N_1 = 1000N$$

Since F_1 is limiting friction,

$$\frac{F_1}{N_1} = \mu = 0.25$$
$$F_1 = 0.25N_1 = 0.25 \times 1000 = 250N$$

$$\sum H = 0$$
$$F_1 - T = 0$$
$$T = F_1 = 250N$$

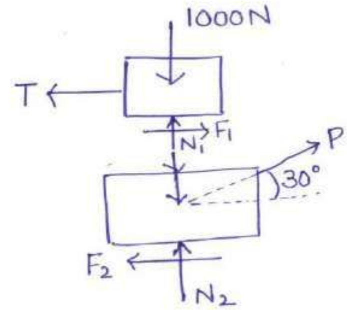
Considering equilibrium of block B,

$$\sum V = 0$$
$$N_2 - 2000 - N_1 = 0$$
$$N_2 = 2000 + N_1 = 2000 + 1000 = 3000N$$

$$\frac{F_2}{N_2} = \mu = \frac{1}{3}$$
$$F_2 = 0.3N_2 = 0.3 \times 1000 = 1000N$$

$$\sum H = 0$$
$$P = F_1 + F_2 = 250 + 1000 = 1250N$$

(b) When P is inclined:



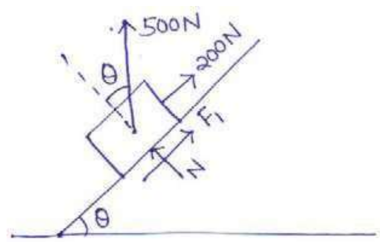
$$\sum V = 0$$
$$N_2 - 2000 - N_1 + P.\sin 30 = 0$$
$$\Rightarrow N_2 + 0.5P = 2000 + 1000$$
$$\Rightarrow N_2 = 3000 - 0.5P$$

From law of friction,

$$F_2 = \frac{1}{3} N_2 = \frac{1}{3}(3000 - 0.5P) = 1000 - \frac{0.5}{3} P$$

$$\sum H = 0$$
$$P \cos 30 = F_1 + F_2$$
$$\Rightarrow P \cos 30 = 250 + \left(1000 - \frac{0.5}{3} P\right)$$
$$\Rightarrow P\left(\cos 30 + \frac{0.5}{3}\right) = 1250$$
$$\Rightarrow P = 1210.43N$$

Problem 2: A block weighing 500N just starts moving down a rough inclined plane when supported by a force of 200N acting parallel to the plane in upward direction. The same block is on the verge of moving up the plane when pulled by a force of 300N acting parallel to the plane. Find the inclination of the plane and coefficient of friction between the inclined plane and theblock.

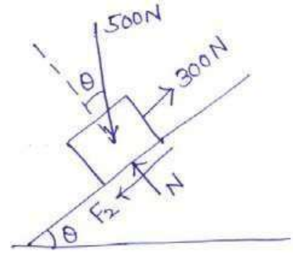


$$\sum V = 0$$
$$N = 500.\cos\theta$$
$$F_1 = \mu N = \mu.500 \cos\theta$$

$\sum H = 0$

$200 + F_1 = 500.\sin\theta$ (1)

$\Rightarrow 200 + \mu.500\cos\theta = 500.\sin\theta$



$\sum V = 0$

$N = 500.\cos\theta$

$F_2 = \mu N = \mu.500.\cos\theta$

$\sum H = 0$

$500\sin\theta + F_2 = 300$ (2)

$\Rightarrow 500\sin\theta + \mu.500\cos\theta = 300$

Adding Eqs. (1) and (2), we get

$500 = 1000.\sin\theta$

$\sin\theta = 0.5$

$\theta = 30^\circ$

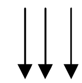
Substituting the value of θ in Eq. 2,

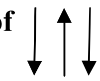
$500\sin 30 + \mu.500\cos 30 = 300$

$$\mu = \frac{50}{500\cos 30} = 0.11547$$

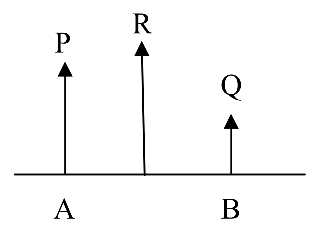
Parallel forces on a plane

Like parallel forces: Coplanar parallel forces when act in the same direction. **Unlike parallel forces:** Coplanar parallel forces when act in different direction. **Resultant of like parallel forces:**



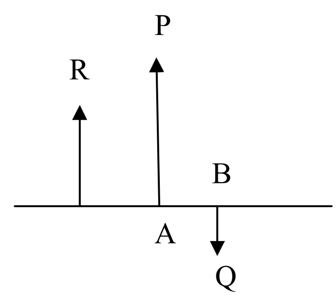


Let P and Q are two like parallel forces act at points A and B. R = P + Q



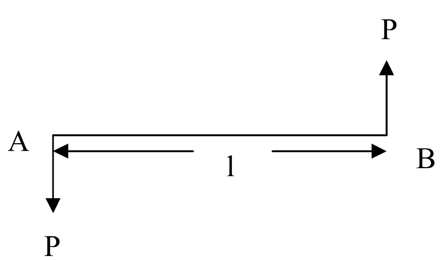
Resultant of unlikeparallelforces:
R = P-Q

R is in the direction of the force havinggreatermagnitude.



Couple:

Two unlike equal parallel forces form a couple.

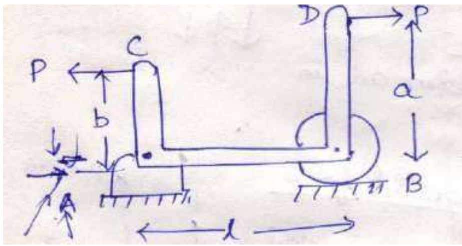


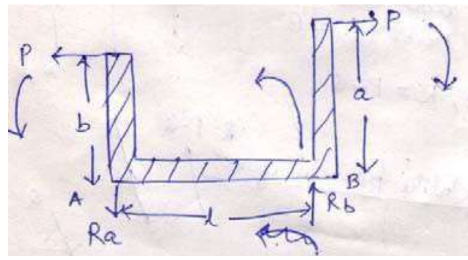
The rotational effect of a couple is measured by its moment.

Moment = P × l

Sign convention: Anticlockwise couple (Positive)
Clockwise couple (Negative)

Problem 1 :A rigid bar CABD supported as shown in figure is acted upon by two equal horizontal forces P applied at C and D. Calculate the reactions that will be induced at the points of support. Assume l = 1.2 m, a = 0.9 m, b =0.6 m.



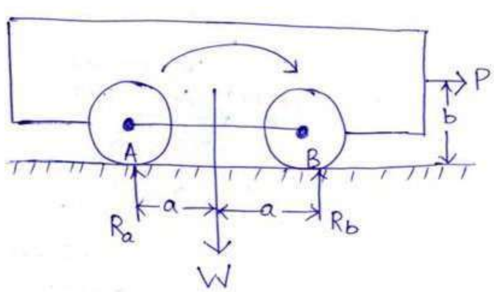


$$\sum V = 0$$
$$R_a = R_b$$

Taking moment about A,

$$R_a = R_b$$
$$R_b \times l + P \times b = P \times a$$
$$\Rightarrow R_b = \frac{P(0.9 - 0.6)}{1.2}$$
$$\Rightarrow R_b = 0.25P(\uparrow)$$
$$\Rightarrow R_a = 0.25P(\downarrow)$$

Problem 2: Owing to weight W of the locomotive shown in figure, the reactions at the two points of support A and B will each be equal to W/2. When the locomotive is pulling the train and the drawbar pull P is just equal to the total friction at the points of contact A and B, determine the magnitudes of the vertical reactions R_a and R_b.



$$\sum V = 0$$
$$R_a + R_b = W$$

Taking moment about B,

$$\sum M_B = 0$$

$$R_a \times 2a + P \times b = W \times a$$

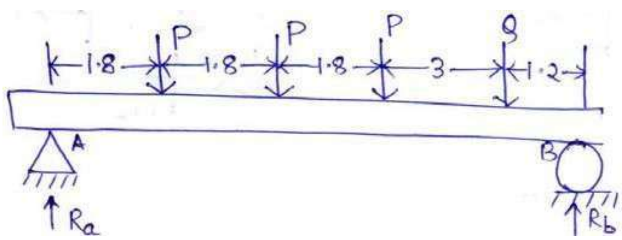
$$\Rightarrow R_a = \frac{W.a - P.b}{2a}$$

$$\therefore R_b = W - R_a$$

$$\Rightarrow R_b = W - \left(\frac{W.a - P.b}{2a}\right)$$

$$\Rightarrow R_b = \frac{W.a + P.b}{2a}$$

Problem 3: The four wheels of a locomotive produce vertical forces on the horizontal girder AB. Determine the reactions R_a and R_b at the supports if the loads P = 90 KN each and Q = 72 KN (All dimensions are in m).



$$\sum V = 0$$

$$R_a + R_b = 3P + Q$$

$$\Rightarrow R_a + R_b = 3 \times 90 + 72$$

$$\Rightarrow R_a + R_b = 342KN$$

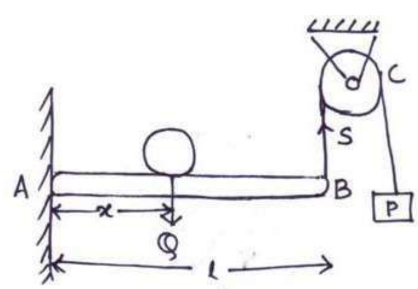
$$\sum M_A = 0$$

$$R_b \times 9.6 = 90 \times 1.8 + 90 \times 3.6 + 90 \times 5.4 + 72 \times 8.4$$

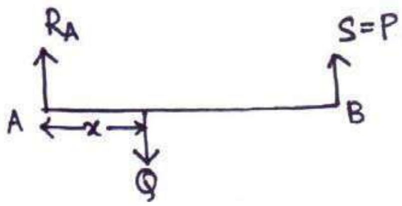
$$\Rightarrow R_b = 164.25KN$$

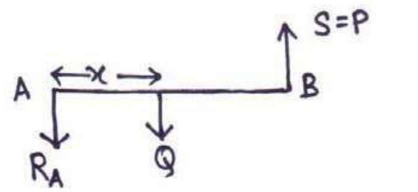
$$\therefore R_a = 177.75KN$$

Problem 4: The beam AB in figure is hinged at A and supported at B by a vertical cord which passes over a frictionless pulley at C and carries at its end a load P. Determine the distance x from A at which a load Q must be placed on the beam if it is to remain in equilibrium in a horizontal position. Neglect the weight of thebeam.



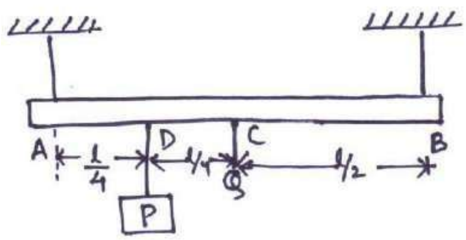
FBD



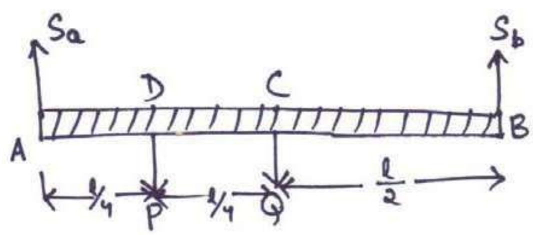


$$\sum M_A = 0$$
$$S \times l = Q \times x$$
$$\Rightarrow x = \frac{P.l}{Q}$$

Problem 5: A prismatic bar AB of weight Q = 44.5 N is supported by two vertical wires at its ends and carries at D a load P = 89 N as shown in figure. Determine the forces S_a and S_b in the two wires.



Q = 44.5 N
P = 89 N



Resolving vertically,

$$\sum V = 0$$
$$S_a + S_b = P + Q$$
$$\Rightarrow S_a + S_b = 89 + 44.5$$
$$\Rightarrow S_a + S_b = 133.5N$$

$$\sum M_A = 0$$

$$S_b \times l = P \times \frac{l}{4} + Q \times \frac{l}{2}$$

$$\Rightarrow S_b = \frac{P}{4} + \frac{Q}{2}$$

$$\Rightarrow S_b = \frac{89}{4} + \frac{44.5}{2}$$

$$\Rightarrow S_b = 44.5$$

$$\therefore S_a = 133.5 - 44.5$$

$$\Rightarrow S_a = 89N$$

Centre of gravity

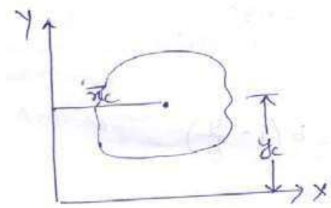
Centre of gravity: It is that point through which the resultant of the distributed gravity force passes regardless of the orientation of the body in space.

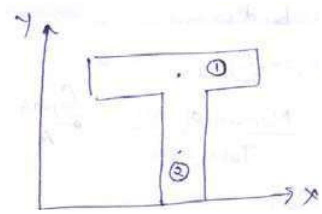
As the point through which resultant of force of gravity (weight) of the bodyacts.

Centroid: Centrroid of an area lies on the axis of symmetry if it exits.

Centre of gravity is applied to bodies with mass and weight and centroid is applied to plane areas.

$$x_c = \sum A_i x_i y_c$$
$$= \sum A_i y_i$$





$$x_c = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2}$$

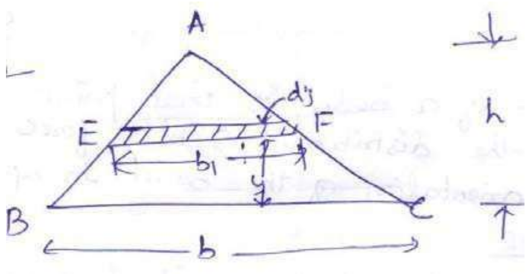
$$y_c = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$$

$$x_c = y_c = \frac{\text{Moment of area}}{\text{Totalarea}}$$

$$x_c = \frac{\int x.dA}{A}$$

$$y_c = \frac{\int y.dA}{A}$$

Problem 1: Consider the triangle ABC of base 'b' and height 'h'. Determine the distance of centroid from the base.



Let us consider an elemental strip of width 'b_1' and thickness 'dy'.

$$\Delta AEF \sim \Delta ABC$$

$$\therefore \frac{b_1}{b} = \frac{h-y}{h}$$

$$\Rightarrow b_1 = b\left(\frac{h-y}{h}\right)$$

$$\Rightarrow b_1 = b\left(1-\frac{y}{h}\right)$$

Area of element EF (dA) = $b_1 \times dy$

$$= b\left(1-\frac{y}{h}\right)dy$$

$$\bar{y} = \frac{\int y.dA}{A}$$

$$= \frac{\int_0^h y.b\left(1-\frac{y}{h}\right)dy}{\frac{1}{2}b.h}$$

$$= \frac{b\left[\frac{y^2}{2}-\frac{y^3}{3h}\right]_0^h}{\frac{1}{2}b.h}$$

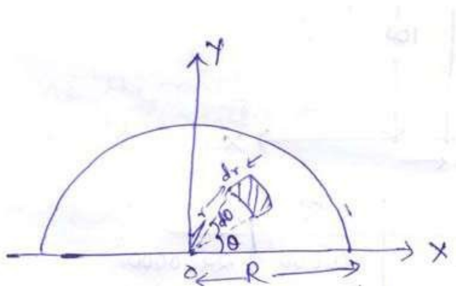
$$= \frac{2}{h}\left[\frac{h^2}{2}-\frac{h^3}{3}\right]$$

$$= \frac{2}{h}\times\frac{h^2}{6}$$

$$= \frac{h}{3}$$

Therefore, y_c is at a distance of h/3 from base.

Problem 2: Consider a semi-circle of radius R. Determine its distance from diametral axis.



Due to symmetry, centroid 'y_c' must lie on Y-axis.

Consider an element at a distance 'r' from centre 'o' of the semicircle with radial width dr.

Area of element = (r.dθ)×dr

Moment of area about x = $\int y.dA$

$$= \int_0^\pi \int_0^R (r.d\theta).dr \times (r.\sin\theta)$$

$$= \int_0^\pi \int_0^R r^2 \sin\theta.dr.d\theta$$

$$= \int_0^\pi \int_0^R (r^2.dr).\sin\theta.d\theta$$

$$= \int_0^\pi \left[\frac{r^3}{3}\right]_0^R .\sin\theta.d\theta$$

$$= \int_0^\pi \frac{R^3}{3}.\sin\theta.d\theta$$

$$= \frac{R^3}{3}\left[-\cos\theta\right]_0^\pi$$

$$= \frac{R^3}{3}\left[1+1\right]$$

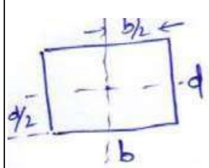
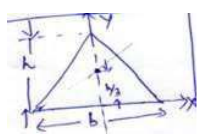
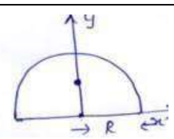
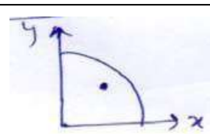
$$= \frac{2}{3}R^3$$

$$y_c = \frac{\text{Moment of area}}{\text{Totalarea}}$$

$$= \frac{\frac{2}{3}R^3}{\frac{\pi R^2}{2}}$$

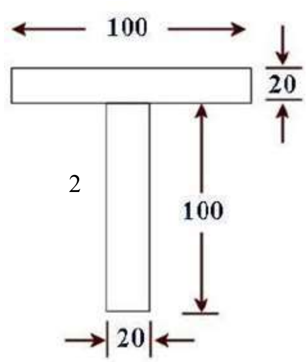
$$= \frac{4R}{3\pi}$$

Centroids of different figures

Shape	Figure	$\overline{x}$	$\overline{y}$	Area
Rectangle		$\frac{b}{2}$	$\frac{d}{2}$	bd
Triangle		0	$\frac{h}{3}$	$\frac{bh}{2}$
Semicircle		0	$\frac{4R}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter circle		$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$	$\frac{\pi r^2}{4}$

Problem 3: Find the centroid of the T-section as shown in figure from the bottom.

1

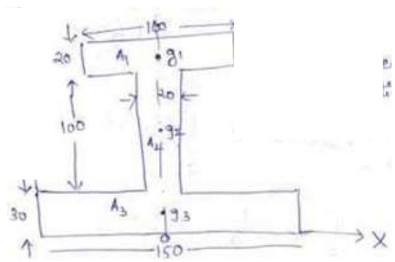


Area (A_i)	x_i	y_i	$A_i x_i$	$A_i y_i$
2000	0	110	10,000	22,0000
2000	0	50	10,000	10,0000
4000			20,000	32,0000

$$\bar{y} = \frac{\sum A_i y_i}{A_i} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{32,0000}{4000} = 8$$

Due to symmetry, the centroid lies on Y-axis and it is at distance of 80 mm from the bottom.

Problem 4: Locate the centroid of the I-section.



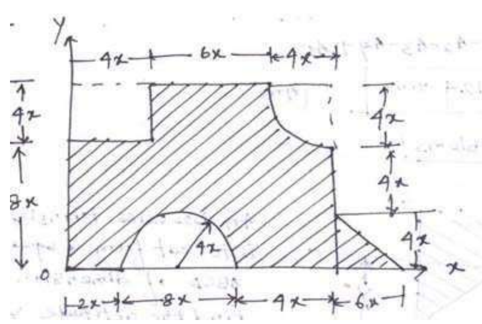
As the figure is symmetric, centroid lies on y-axis. Therefore, $\bar{x} = 0$

Area (A_i)	x_i	y_i	$A_i x_i$	$A_i y_i$
2000	0	140	0	280000
2000	0	80	0	160000
4500	0	15	0	67500

$$\bar{y} = \frac{\sum A_i y_i}{A_i} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3} = 59.71\,mm$$

Thus, the centroid is on the symmetric axis at a distance 59.71 mm from the bottom.

Problem 5: Determine the centroid of the composite figure about x-y coordinate. Take x = 40 mm.



A_1 = Area of rectangle = 12x.14x=168x^2

A_2 = Area of rectangle to be subtracted = 4x.4x = 16 x^2

A_3 = Area of semicircle to be subtracted = $\frac{\pi R^2}{2} = \frac{\pi (4x)^2}{2}$ $= 25.13x^2$

A_4 = Area of quatercircle to be subtracted = $\frac{\pi R^2}{\ } = \frac{\pi (4x)^2}{\ }$ $= 12.56x^2$

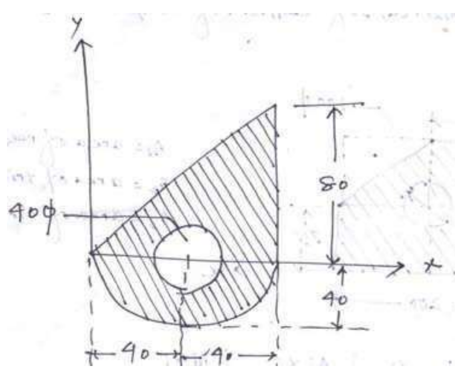
A_5 = Area of triangle $= \frac{1}{2} \times 6x \times 4x = 12x^2$

Area (A_i)	**x_i**	**y_i**	**$A_i x_i$**	**$A_i y_i$**
A_1 = 268800	7x = 280	6x =240	75264000	64512000
A_2 = 25600	2x = 80	10x=400	2048000	10240000
A_3 = 40208	6x =240	$\frac{4 \times 4x}{3\pi}$ =67.906	9649920	2730364.448
A_4 = 20096	$10x + \left(4x - \frac{4\times 4x}{3\pi}\right)$ = 492.09	$8x + \left(4x - \frac{4\times 4x}{3\pi}\right)$ = 412.093	9889040.64	8281420.926
A_5 = 19200	$14x + \frac{6x}{3} = 16x$ = 640	$\frac{4x}{3}$ = 53.33	12288000	1023936

$$\bar{x} = \frac{A_1x_1 - A_2x_2 - A_3x_3 - A_4x_4 + A_5x_5}{A_1 - A_2 - A_3 - A_4 + A_5} = 326.404mm$$

$$\bar{y} = \frac{A_1y_1 - A_2y_2 - A_3y_3 - A_4y_4 + A_5y_5}{A_1 - A_2 - A_3 - A_4 + A_5} = 219.124mm$$

Problem 6: Determine the centroid of the following figure.



A_1 = Area of triangle = $\frac{1}{2} \times 80 \times 80 = 3200m^2$

A_2 = Area of semicircle= $\frac{\pi d^2}{8} = \frac{\pi R^2}{2}$ 2513.274m

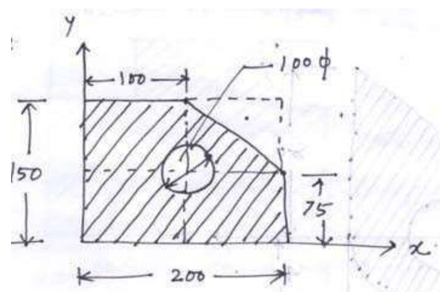
A_3 = Area of semicircle $\frac{\pi D^2}{\ } = 1256.64m^2$

Area (A_i)	x_i	y_i	$A_i\,x_i$	A_iy_i
3200	2×(80/3)=53.33	80/3 = 26.67	170656	85344
2513.274	40	$\frac{-4\times40}{3\pi}=-16.97$	100530.96	-42650.259
1256.64	40	0	50265.6	0

$$x_c=\frac{A_1x_1+A_2x_2-A_3x_3}{A_1+A_2+A_3}=49.57mm$$

$$y_c=\frac{A_1y_1+A_2y_2-A_3y_3}{A_1+A_2-A_3}=9.58mm$$

Problem 7: Determine the centroid of the following figure.



A_1 = Area of the rectangle
A_2 = Area of triangle
A_3 = Area of circle

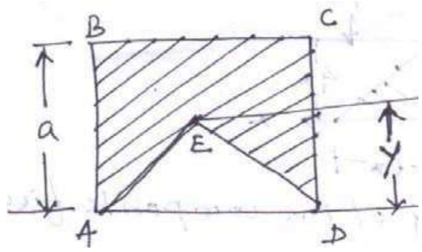
Area (A_i)	x_i	y_i	$A_i\,x_i$	A_iy_i
30,000	100	75	3000000	2250000
3750	100+200/3 = 166.67	75+150/3 =125	625012.5	468750
7853.98	100	75	785398	589048.5

$$x_c=\frac{\sum A_ix_i}{\sum A_i}=\frac{A_1x_1-A_2x_2-A_3x_3}{A_1-A_2-A_3}=86.4mm$$

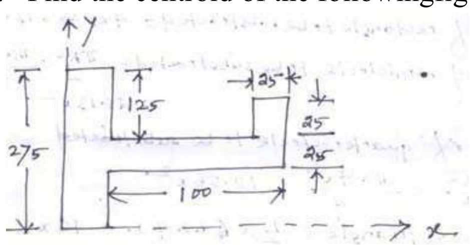
$$y_c=\frac{\sum A_iy_i}{\sum A_i}=\frac{A_1y_1-A_2y_2-A_3y_3}{A_1-A_2-A_3}=64.8mm$$

Numerical Problems (Assignment)

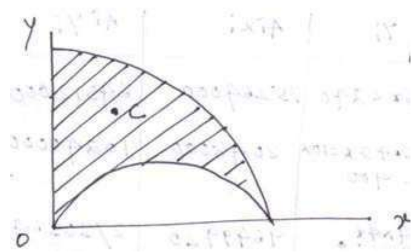
1. An isosceles triangle ADE is to cut from a square ABCD of dimension ‘a’. Find the altitude ‘y’ of the triangle so that vertex E will be centroid of remaining shadedarea.



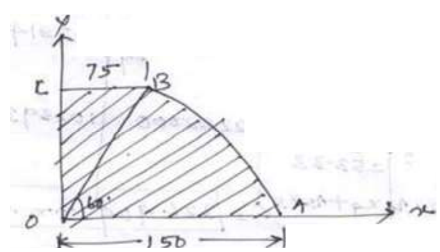
2. Find the centroid of the followingfigure.



3. Locate the centroid C of the shaded area obtained by cutting a semi-circle of diameter ‘a’ from the quadrant of a circle of radius‘a’.



4. Locate the centroid of the compositefigure.



Truss/ Frame: A pin jointed frame is a structure made of slender (cross-sectional dimensions quite small compared to length) members pin connected at ends and capable of taking load at joints.

Such frames are used as roof trusses to support sloping roofs and as bridge trusses to support deck.

Plane frame: A frame in which all members lie in a single plane is called plane frame. They are designed to resist the forces acting in the plane of frame. Roof trusses and bridge trusses are the example of plane frames.

Space frame: If all the members of frame do not lie in a single plane, they are called as space frame. Tripod, transmission towers are the examples of spaceframes.

Perfect frame: A pin jointed frame which has got just sufficient number of members to resist the loads without undergoing appreciable deformation in shape is called a perfect frame. Triangular frame is the simplest perfect frame and it has 03 joints and 03members.

It may be observed that to increase one joint in a perfect frame, two more members are required. Hence, the following expression may be written as the relationship between number of joint j, and the number of members m in a perfect frame.

$$m = 2j - 3$$

(a) When LHS = RHS, Perfectframe.
(b) When LHS<RHS, Deficientframe.
(c) When LHS>RHS, Redundantframe.

<u>Assumptions</u>

The following assumptions are made in the analysis of pin jointed trusses:

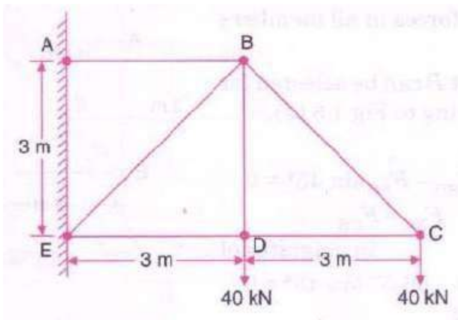
1. The ends of the members are pin jointed(hinged).
2. The loads act only at thejoints.
3. Self weight of the members isnegligible.

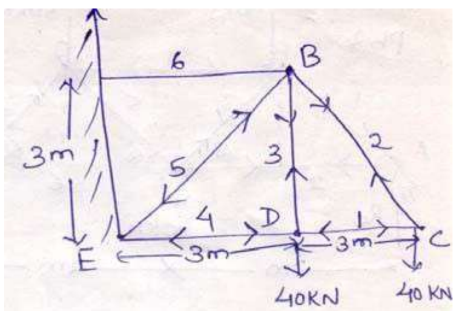
Methods of analysis

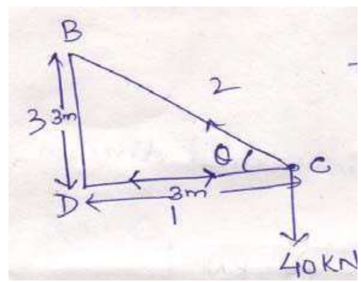
1. Method ofjoint
2. Method ofsection

Problems on method of joints

Problem 1: Find the forces in all the members of the truss shown in figure.



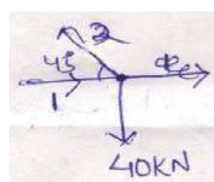




$\tan\theta = 1$

$\Rightarrow \theta = 45^\circ$

<u>Joint C</u>



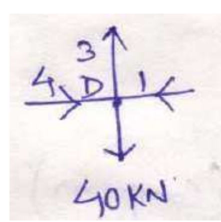
$S_1 = S_2 \cos 45$

$\Rightarrow S_1 = 40KN$(Compression)

$S_2 \sin 45 = 40$

$\Rightarrow S_2 = 56.56KN$ (Tension)

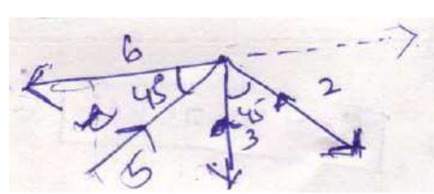
<u>Joint D</u>



$S_3 = 40KN$ (Tension)

$S_1 = S_4 = 40KN$ (Compression) <u>Joint</u>

<u>B</u>



Resolving vertically,

$\sum V = 0$

$S_5 \sin 45 = S_3 + S_2 \sin 45$

$\Rightarrow S_5 = 113.137 KN$ (Compression)

Resolving horizontally,

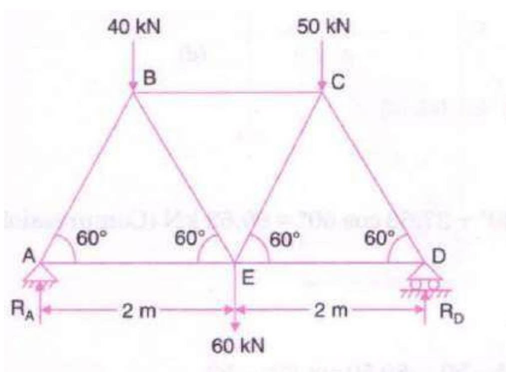
$\sum H = 0$

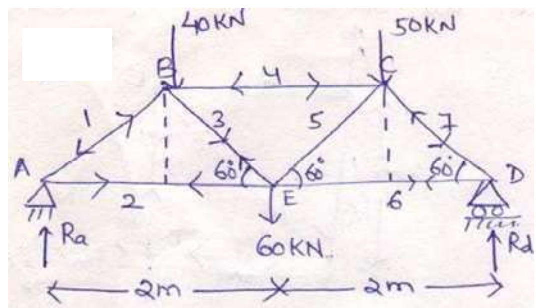
$S_6 = S_5 \cos 45 + S_2 \cos 45$

$\Rightarrow S_6 = 113.137 \cos 45 + 56.56 \cos 45$

$\Rightarrow S_6 = 120 KN$ (Tension)

Problem 2: Determine the forces in all the members of the truss shown in figure and indicate the magnitude and nature of the forces on the diagram of the truss. All inclined members are at 60˚ to horizontal and length of each member is2m.





Taking moment at point A,

$\sum M_A = 0$

$R_d \times 4 = 40 \times 1 + 60 \times 2 + 50 \times 3$

$\Rightarrow R_d = 77.5 KN$

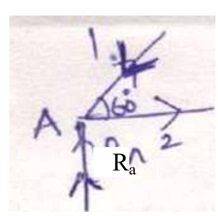
Now resolving all the forces in vertical direction,

$\sum V = 0$

$R_a + R_d = 40 + 60 + 50$

$\Rightarrow R_a = 72.5 KN$

Joint A



$\sum V = 0$

$\Rightarrow R_a = S_1 \sin 60$

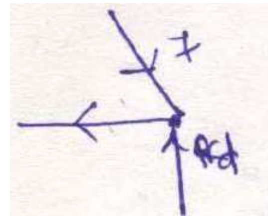
$\Rightarrow S_1 = 83.72 KN$ (Compression)

$\sum H = 0$

$\Rightarrow S_2 = S_1 \cos 60$

$\Rightarrow S_1 = 41.86KN$(Tension)

Joint D



$\sum V = 0$

$S_7 \sin 60 = 77.5$

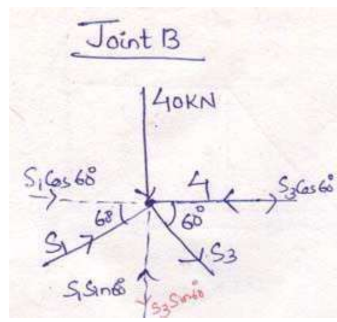
$\Rightarrow S_7 = 89.5KN$ (Compression)

$\sum H = 0$

$S_6 = S_7 \cos 60$

$\Rightarrow S_6 = 44.75KN$ (Tension)

Joint B



$\sum V = 0$

$S_1 \sin 60 = S_3 \cos 60 + 40$

$\Rightarrow S_3 = 37.532KN$ (Tension)

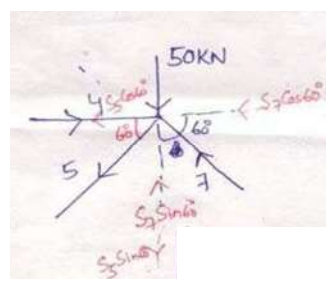
$\sum H = 0$

$S_4 = S_1 \cos 60 + S_3 \cos 60$

$\Rightarrow S_4 = 37.532 \cos 60 + 83.72 \cos 60$

$\Rightarrow S_4 = 60.626KN$ (Compression)

Joint C



$\sum V = 0$

$S_5 \sin 60 + 50 = S_7 \sin 60$

$\Rightarrow S_5 = 31.76KN$ (Tension)

Plane Truss (Method of Section)

In case of analysing a plane truss, using method of section, after determing the support reactions a section line is drawn passing through not more than three members in which forces are unknown, such that the entire frame is cut into two separate parts.

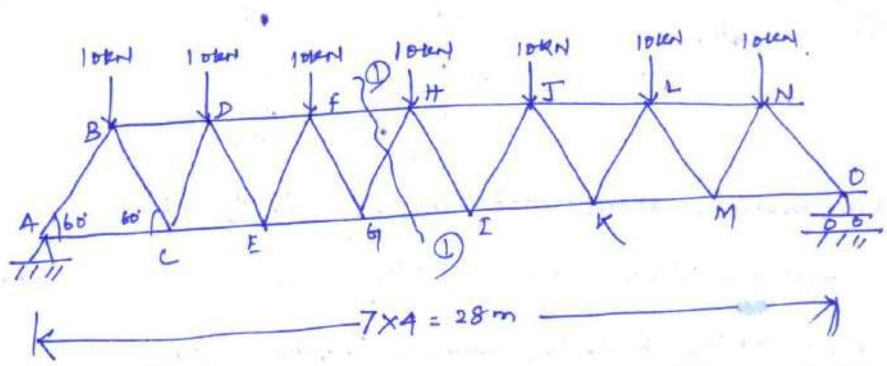
~~Ed.~~ Each part should be in equilibrium under the action of loads, reactions and the forces in the members.

Method of section is preferred for the following cases:

(i) analysis of large truss in which forces in only few members are required

(ii) If method of joint fails to start or proceed with analysis for not getting a joint with only two unknown forces.

Example 1.

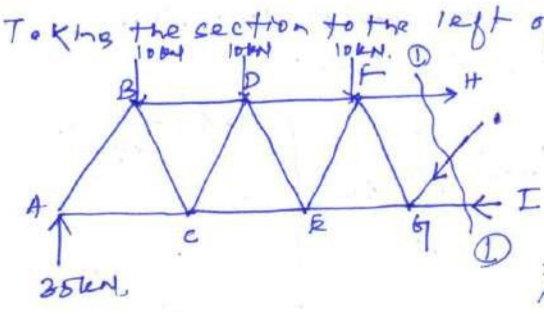


Determine the forces in the members FH, HG, and GI in the truss

Due to symmetry $R_A = R_O = \frac{1}{2} \times$ total downward load

$= \frac{1}{2} \times 70 = \boxed{35\,kN.}$

Taking the section to the left of the cut.



Taking moment about G

$\Sigma M_G = 0.$

$F_{FH} \times 4\sin 60 + 35 \times 12$

$= 10 \times 2 + 10 \times 6 + 10 \times 10$

$\Rightarrow F_{FH} = \dfrac{(20+60+100) - 420}{4\sin 60°}$

$= -69.28\ kN.$

Negative sign indicates that direction should have opposite i.e it is compressive in nature.

Now Resolving all the forces vertically $\Sigma Y = 0$

$$10 + 10 + 10 + F_{GH} \sin 60 = 35$$

$$\Rightarrow F_{GH} = \frac{35 - 30}{\sin 60^\circ}$$

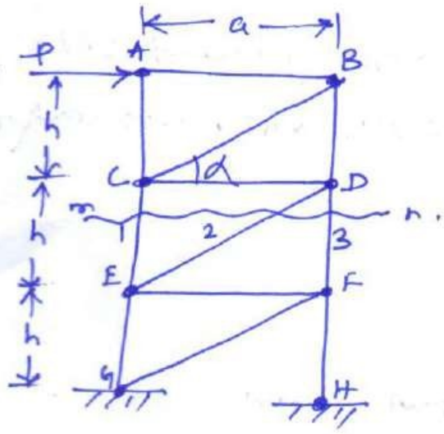
$$\Rightarrow \boxed{F_{GH} = 5.78 \text{ kN.}} \text{ (compressive)}$$

Resolving all the forces horizontally $\Sigma X = 0$.

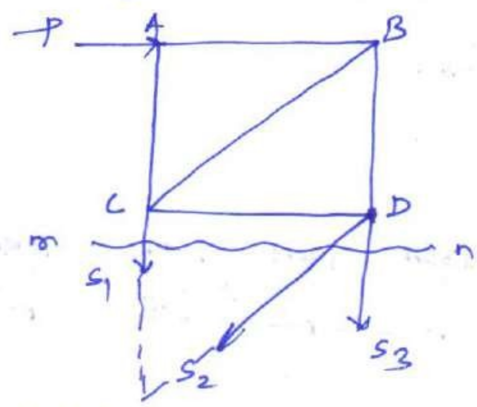
$$F_{FH} + F_{GH} \cos 60 = F_{GI}$$

$$\Rightarrow F_{GI} = 69.28 + 5.78 \cos 60^\circ = \boxed{72.17 \text{ kN.}} \text{ (tension)}$$

Q.2



Using method of sections determine the axial forces in bars 1, 2 and 3.



Taking moment about ~~poi~~ joint D $\Sigma M_D = 0$.

$$S_1 \times a = P \times h \Rightarrow \boxed{S_1 = \frac{Ph}{a}} \quad \text{— (1)} \quad \text{(tension)}$$

Similarly taking E as the moment centre $\Sigma M_E = 0$

$$S_3 \times a + P \times 2h = 0$$

$$\Rightarrow \boxed{S_3 = \frac{-2Ph}{a}}$$

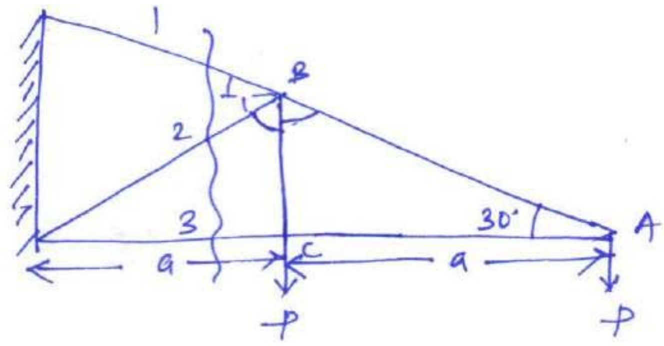
(−ve sign indicates direction of force will be opposite and it will be compressive in nature).

Resolving all the forces horizontally. $\Sigma X = 0$.

$$S_2 \cos\alpha = P$$

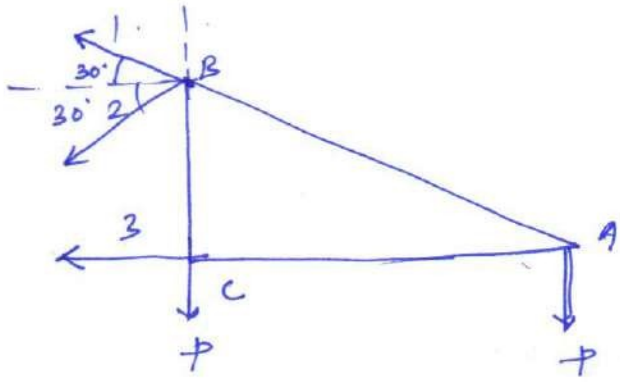
$$\Rightarrow S_2 = \frac{P}{\cos\alpha} = \boxed{\frac{P\sqrt{a^2 + h^2}}{a}} \text{ (Ans)}$$

$$\cos\alpha = \frac{a}{\sqrt{a^2 + h^2}}$$



$\frac{BC}{AC} = \tan 30°$

$\Rightarrow BC = a \tan 30 = \boxed{0.578a}$



$\Sigma M_B = 0.$

$S_3 \times 0.578a + P \times a = 0$

$\Rightarrow S_3 = \frac{-Pa}{0.578a} = -1.73P$

(-ve sign indicates direction is opposite and it is compressive in nature)

Resolving vertically $\Sigma Y = 0$

$S_1 \sin 30 = 2P + S_2 \sin 30$

$\Rightarrow S_1 = \frac{2P + S_2/2}{\sin 30} = (4P + S_2)$ — (2)

Now resolving horizontally $\Sigma X = 0.$

$S_1 \cos 30 + S_2 \cos 30 = 1.73P$

$\Rightarrow (4P + S_2) \times \frac{\sqrt{3}}{2} + S_2 \frac{\sqrt{3}}{2} = 1.73P$

$\Rightarrow 2\sqrt{3}P + \frac{\sqrt{3}}{2} S_2 + \frac{\sqrt{3}}{2} S_2 = 1.73P$

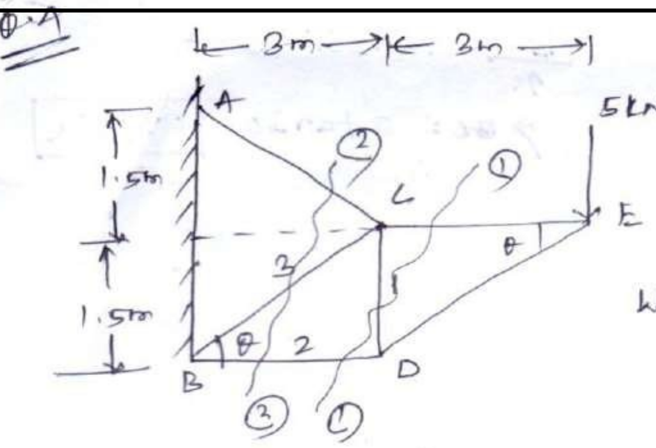
$\Rightarrow \frac{\sqrt{3}}{2} S_2 = 1.73P - 2\sqrt{3}P$

$= -1.73P$

$\Rightarrow S_2 = \frac{-1.73P}{\sqrt{3}} = \boxed{-P}$ (-ve sign indicates the direction is opposite and it is compressive)

Now $S_1 = 4P - P = \boxed{3P}$ (tension)

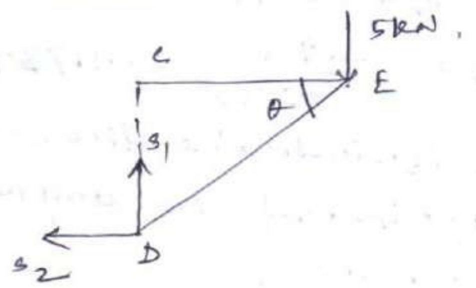
Q.4



Using method of sections, find axial forces in each bar 1, 2 and 3 of the plane truss.

We have $\tan\theta = \left(\frac{1.5}{3}\right) \Rightarrow \boxed{\theta = 26.56^\circ}$

considering section 1-1



Resolving vertically, $\Sigma Y = 0$

$\boxed{S_1 = 5kN}$ (tension)

Now taking moment about C

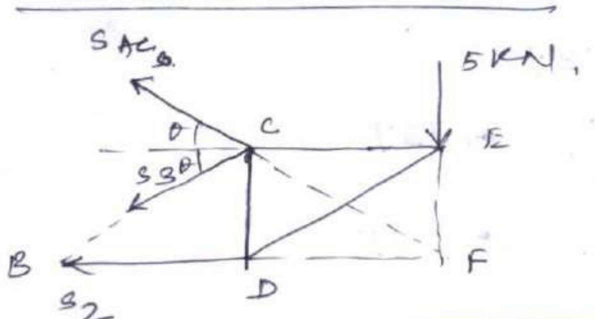
$S_2 \times 1.5 + 5 \times 3 = 0$

$\Rightarrow S_2 = -10kN$

-ve sign indicates direction should have been opposite

$\boxed{S_2 = 10kN}$ (compression)

considering section 2-2

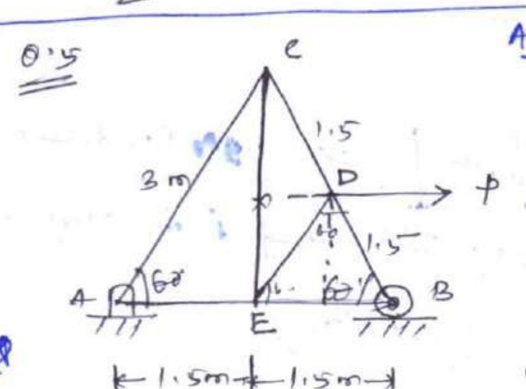


Taking moment about F

$\Sigma M_F = 0$

$\Rightarrow \boxed{S_3 = 0}$

Q.5



Assignment

Using method of joint and method of section find the axial force in the bar x.

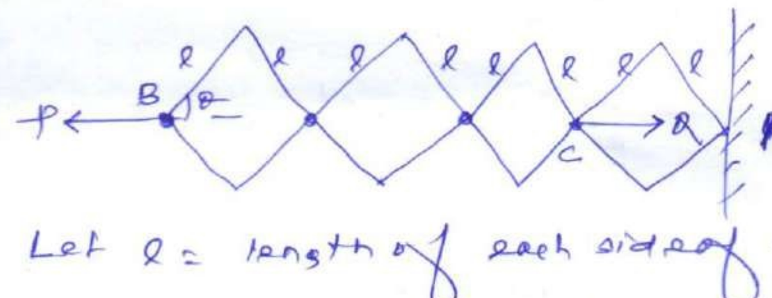
Method of Joint

considering the whole structure and taking moment about A $\Sigma M_A = 0$.

$R_B \times 3 = P \times 1.5 \sin 60$

$\Rightarrow \boxed{R_B = \frac{\sqrt{3}}{4} P.}$

Q.1 (6.3) Calculate the relation betn active forces P and Q for equilibrium of system of bars. The bars are so arranged that they form identical rhombuses.



Let l = length of each side of bar.

θ = angle made by each side of the rhombus

Distance of P from fixed point A = $6l\cos\theta$

" Q " $= 2l\cos\theta$

Let the virtual displacement of P is B-B'

$B\text{-}B' = dx_1 = -6l\sin\theta\, d\theta$

Similarly the virtual displacement of Q is C-C'

$= dx_2 = -2l\sin\theta\, d\theta$

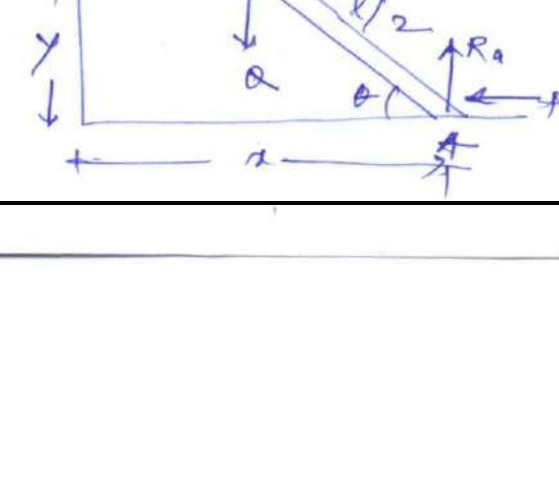
Applying principle of virtual work $\Sigma W = 0$

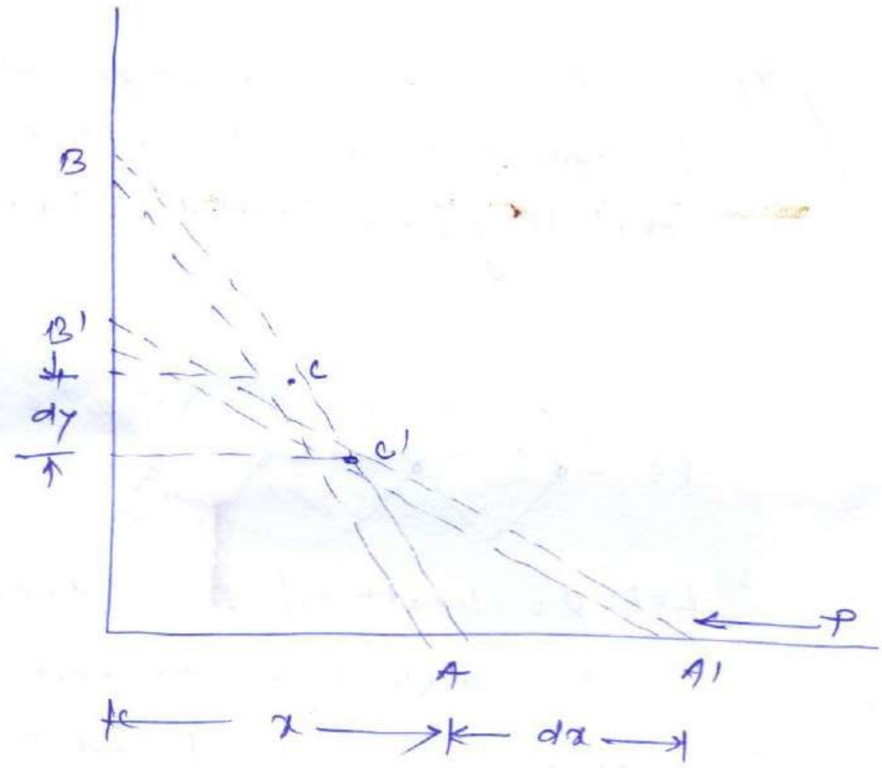
$P \cdot dx_1 = Q \cdot dx_2$

$\Rightarrow P \cdot (6l\sin\theta\, d\theta) = Q(2l\sin\theta\, d\theta)$

$\Rightarrow \boxed{P = \frac{Q}{3}}$ (Ans).

Q.2 A prismatic bar AB of length l and wt. Q stands in a vertical plane and is supported by smooth surfaces at A and B. Using principle of virtual work find the magnitude of horizontal force P applied at A if the bar is in equilibrium.





Let the horizontal distance of P from D is x

$x = l\cos\theta$

$AA' = dx = -l\sin\theta\, d\theta$

vertical distance of Q from D is y

$y = \frac{l}{2}\sin\theta$

$CC' = dy = \frac{l}{2}\cos\theta\, d\theta$

Normal reactions R_a and R_b have no work along the planes.

Applying principle of virtual work $\Sigma W = 0$

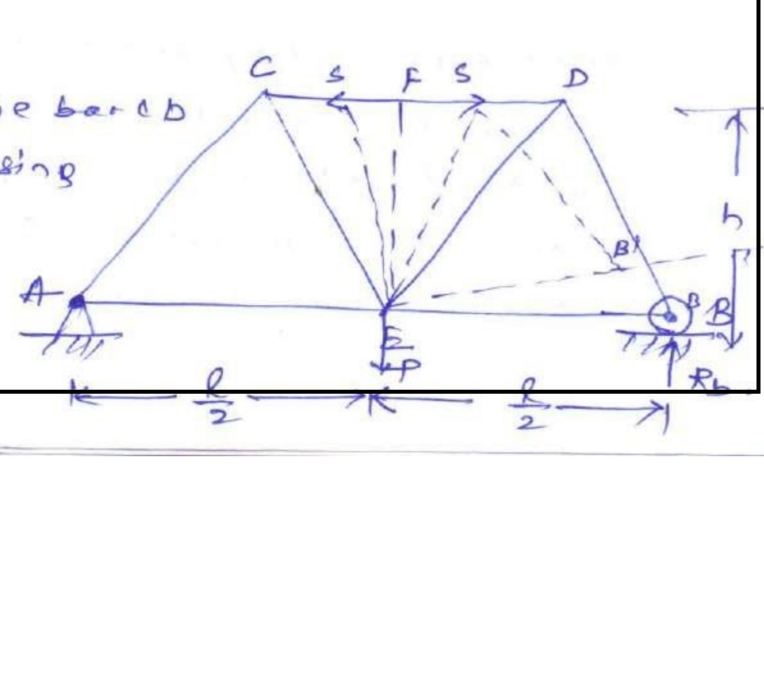
$P\,dx = Q\,dy$

$P l \sin\theta\, d\theta = Q\frac{l}{2}\cos\theta\, d\theta$

$\Rightarrow \boxed{P = \frac{Q}{2}\cdot\cot\theta}$

Q.3 (6.14)

Find axial forces in the bar CD of the simple truss by using method of virtual work.



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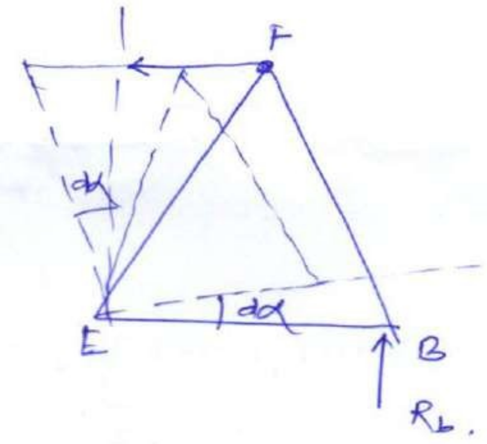
2

Let S be the compressive force in bar CD.

consider the part EBDF of the truss under the action of force R_b, P and S

Keeping E fixed and giving EB an angular displacement $d\alpha$



$\Sigma W = 0$.

$R_b \times BB' = S \times FF'$

$BB' = \frac{l}{2} d\alpha$

$FF' = h\, d\alpha$

$R_b \times \frac{l}{2} d\alpha = S \times h\, d\alpha$

$\Rightarrow \boxed{S = \frac{R_b l}{2h}}$ —— (1)

Now considering whole frame as equilibrium body $\Sigma y = 0$.

$R_a + R_b = P$.

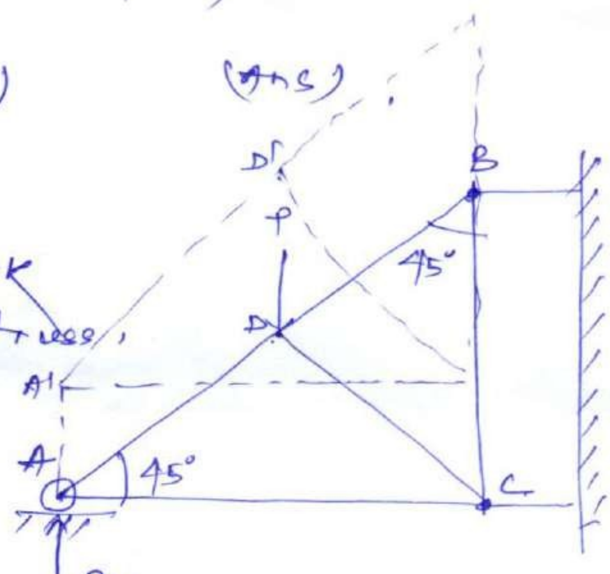
$R_b \cdot l = P \cdot \frac{l}{2} \Rightarrow \boxed{R_b = \frac{P}{2}}$ —— (2)

Substituting the value of R_b in eq. (1)

$\boxed{S = \frac{Pl}{4h}}$ —— (3) (Ans)

Q.4 (6.15)

Using principle of virtual work find reactions R_a for the truss.



Let the truss is virtual displaced by an amount dy

$\Sigma W = 0$.

$R_a \times AA' = P \times DD'$

where $AA' = DD' = dy$

$\Rightarrow \boxed{R_a = P}$

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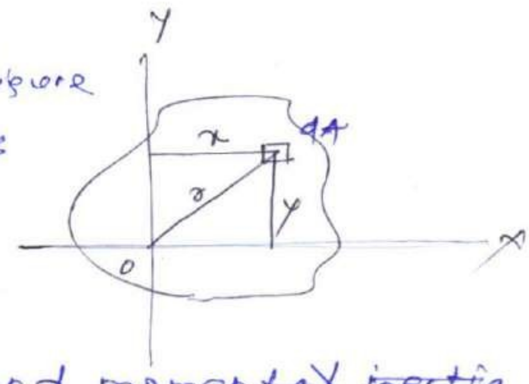
Moment of Inertia of Plane figures

The moment of inertia of any plane figure with respect to x and y axes in its plane are expressed as



$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$

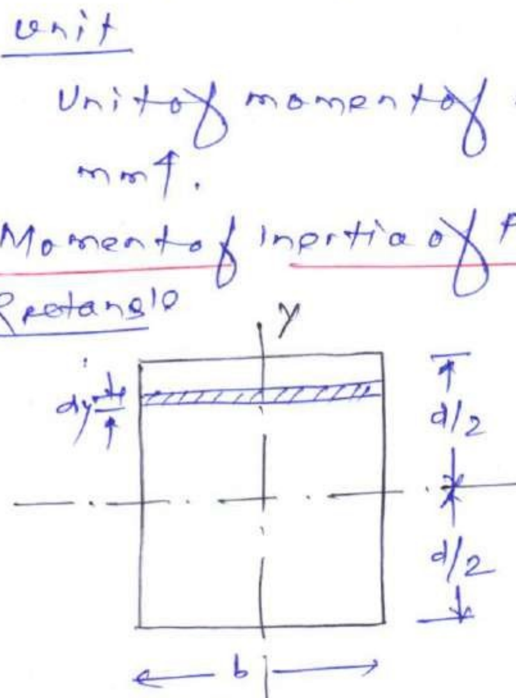
- I_{xx} and I_{yy} are also known as second moment of ~~inertia~~ area about the axes as it is distance is squared from corresponding axis.

Unit

Unit of moment of inertia of area is expressed as m^4 or mm^4.

Moment of Inertia of Plane figures:-

(i) Rectangle



Considering a rectangle of width b and depth d.
Moment of inertia about centroidal axis x-x parallel to the short side i.e b
Now considering an elementary strip of width dy

Moment of inertia of the elemental strip about centroidal axis xx is

$$I_{xx} = y^2 dA$$
$$= y^2 b\,dy$$

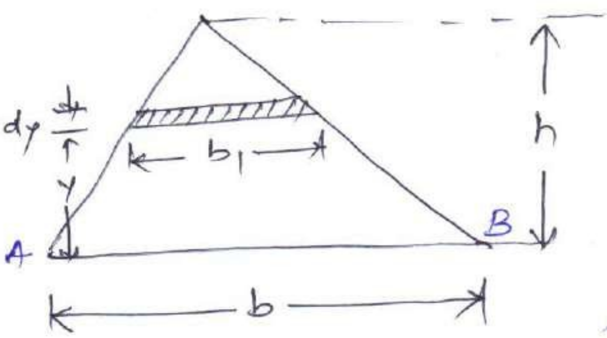
So moment of inertia of entire ~~figure~~ area

$$I_{xx} = \int_{-d/2}^{d/2} y^2 b\,dy = b\left|\frac{y^3}{3}\right|_{-\frac{d}{2}}^{d/2} = b\left[\frac{d^3}{24} + \frac{d^3}{24}\right]$$

$$\Rightarrow \boxed{I_{xx} = \frac{bd^3}{12}}$$

Similarly ~~moment at~~ $\boxed{I_{yy} = \frac{db^3}{12}}$

(ii) Triangle :- (Moment of inertia of a triangle about it's b



Consider a small elementary strip at a distance y from the base of thickness dy. Let dA is the area of strip

$dA = b_1 dy$

And $b_1 = \frac{(h-y)}{h} \times b$.

Moment of inertia of strip about base AB

$= y^2 dA = y^2 b_1 dy$

$= y^2 \frac{(h-y)}{h} b\,dy$

∴ Moment of inertia of the triangle about AB

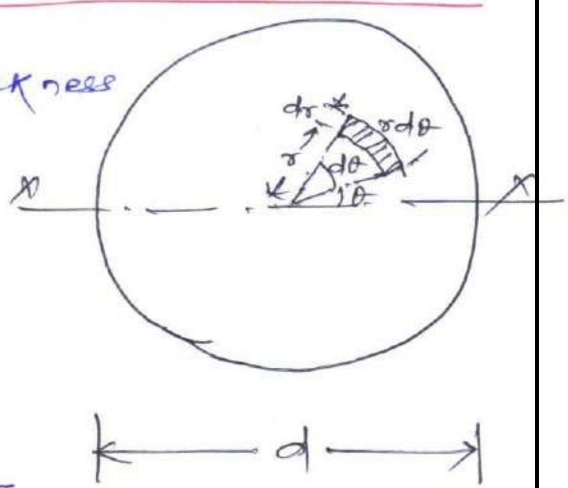
$$I_{AB} = \int_0^h \frac{y^2(h-y)\,b\,dy}{h} = \int_0^h \left(y^2 - \frac{y^3}{h}\right) b\,dy$$

$$= b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = b\left[\frac{h^3}{3} - \frac{h^4}{4h}\right]$$

$$= b\left[\frac{h^3}{3} - \frac{h^3}{4}\right] = \frac{bh^3}{12}$$

$$\Rightarrow \boxed{I_{AB} = \frac{bh^3}{12}}$$

(iii) Moment of inertia of a circle about it's centroidal axis



Considering an elementary strip of thickness dr, the side of strip rdθ.

moment of inertia of strip about xx

$= y^2 dA$

$= (r\sin\theta)^2\, r\,d\theta\,dr$

$= r^3 \sin^2\theta\, d\theta\, dr$

∴ Moment of inertia of circle about xx axis

$$I_{xx} = \int^R \int^{2\pi} r^3 \sin^2\theta\, d\theta\, dr$$

$$= \int_0^R \int_0^{2\pi} r^3 \left(\frac{1-\cos 2\theta}{2}\right) d\theta\, dr$$

$$= \int_0^R \frac{r^3}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} dr$$

$$= \int_0^R \frac{r^3}{2}\left(2\pi - \frac{\sin 4\pi}{2}\right) dr$$

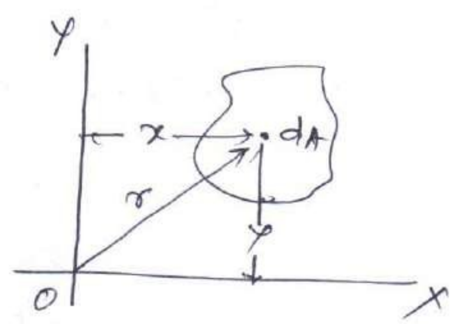
$$= \left[\frac{r^4}{8}\right]_0^R \left[2\pi - 0\right]$$

$$= \frac{R^4}{8} 2\pi = \frac{\pi R^4}{4}$$

$$\Rightarrow \boxed{I_{xx} = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}} \qquad \left(\because R = \frac{D}{2}\right)$$

Polar moment of inertia:-

Moment of inertia about an axis perpendicular to the plane of area is called polar moment of inertia it may denoted as J or I_{zz}



$$\boxed{I_{zz} = \Sigma r^2 dA}$$

Radius of Gyration:-

Radius of gyration may be defined by a relation

$$\boxed{K = \sqrt{\frac{I}{A}}}$$

where K = radius of gyration
I = moment of inertia
A = cross-sectional area

so, we can have the following relations

$$\boxed{\begin{aligned} K_{xx} &= \sqrt{\frac{I_{xx}}{A}} \\ K_{yy} &= \sqrt{\frac{I_{yy}}{A}} \\ K_{AB} &= \sqrt{\frac{I_{AB}}{A}} \end{aligned}}$$

Theorems of Moment of inertia

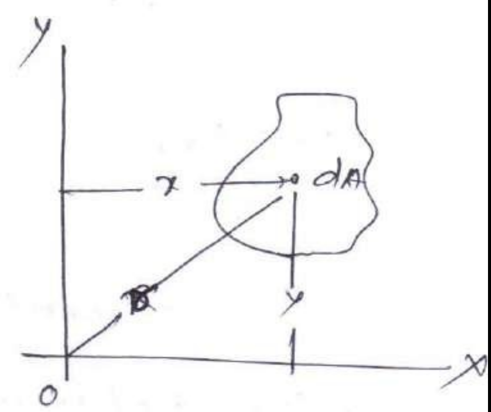
There are two theorems of moment of inertia

(a) Perpendicular axis theorem

(b) parallel axis theorem.

Perpendicular axis theorem:-

Moment of inertia of an area about an axis $\perp$r to it's plane at any point O is equal to the sum of moments of inertia about any two mutually perpendicular axis through the same point O and lying in the plane of area.



$$I_{zz} = I_{xx} + I_{yy}$$

$$I_{zz} = \Sigma r^2 dA$$

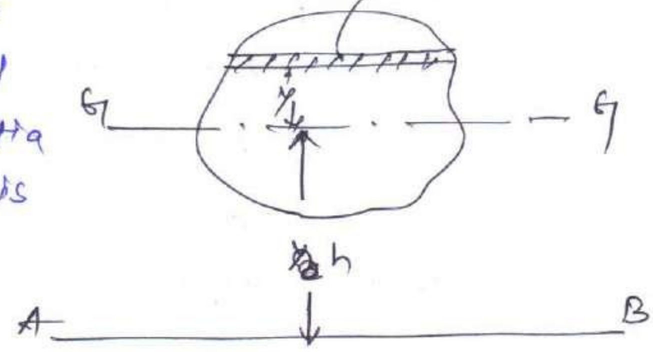
$$= \Sigma(x^2+y^2)\, dA$$

$$= \Sigma x^2 dA + \Sigma y^2 dA$$

$$\Rightarrow \boxed{I_{zz} = I_{xx} + I_{yy}}$$

Parallel axis theorem:-

Moment of inertia about an axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroidal axis and the product of area and square of the distance betn the two parallel axes.



$$\boxed{I_{AB} = \cancel{I_{xx}}\ I_{GG} + Ah^2}$$

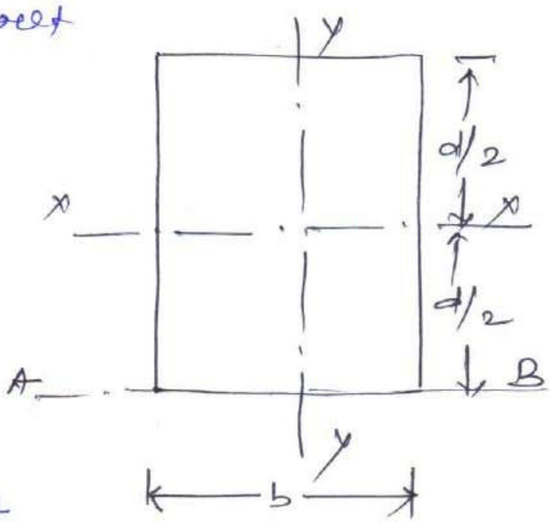
Moment of inertia of standard sections:-

(i) Moment of inertia of a rectangle about its centroidal axis xx

$$I_{xx} = \frac{bd^3}{12}$$

Similarly moment of inertia about its centroidal axis yy

$$I_{yy} = \frac{db^3}{12}$$



Now moment of inertia of rectangle about it's base AB can be obtained by applying parallel axis theorem

$$I_{AB} = I_{xx} + Ah^2$$

$$= \frac{bd^3}{12} + (bd)\left(\frac{d}{2}\right)^2$$

$$= \frac{bd^3}{12} + \frac{bd^3}{4}$$

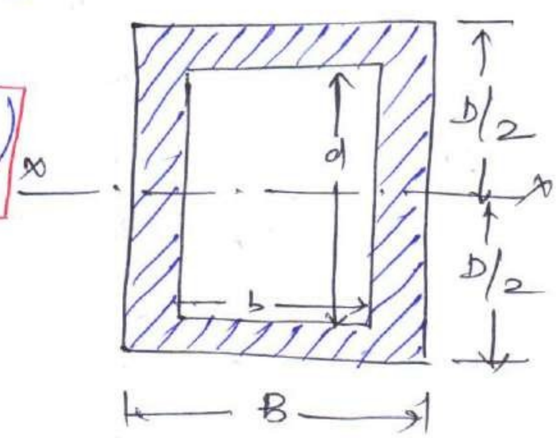
$$= \frac{3bd^3 + bd^3}{12} = \frac{bd^3}{3}$$

$$\Rightarrow \boxed{I_{AB} = \frac{bd^3}{3}}$$

(ii) Moment of inertia of a hollow rectangular section:-

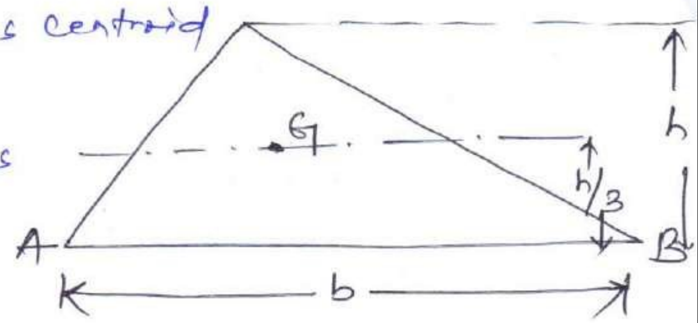
Moment of inertia of hollow rectangular section

$$\boxed{I_{xx} = \frac{BD^3}{12} - \frac{bd^3}{12} = \frac{1}{12}\left(BD^3 - bd^3\right)}$$



(iii) Moment of inertia of ~~triangle about it's base~~ centroidal

Moment of inertia of triangle about it's base
= moment of inertia about its centroid
+ Ah^2
(using parallel axis theorem)



$\Rightarrow I_{AB} = I_{xx} + Ah^2$

$$\Rightarrow \frac{bh^3}{12} = I_{xx} + \frac{1}{2} b \times h \times \left(\frac{h}{3}\right)^2$$

$$= I_{xx} + \frac{bh^3}{18}$$

$$\Rightarrow I_{xx} = \frac{bh^3}{12} - \frac{bh^3}{18} = \frac{3bh^3 - 2bh^3}{36} = \frac{bh^3}{36}$$

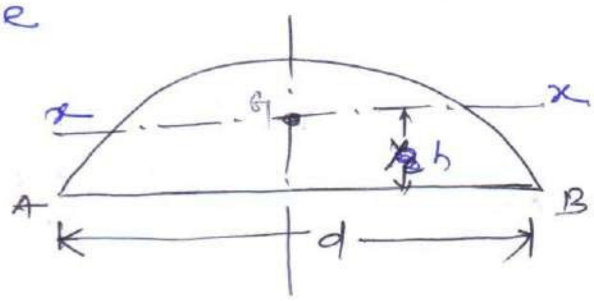
$$\Rightarrow \boxed{I_{xx} = \frac{bh^3}{36}}$$

(iv) Moment of inertia of semicircle

(a) about diametral axis

Moment of inertia of semicircle about $AB = \frac{1}{2} \frac{\pi d^4}{64}$

$$= \boxed{\frac{\pi d^4}{128}}$$



(b) about centroidal axis xx

$$\boxed{h = \frac{4R}{3\pi} = \frac{2d}{3\pi}}$$

area $A = \frac{1}{2} \frac{\pi d^2}{4} = \frac{\pi d^2}{8}$

Using parallel axis theorem

$I_{AB} = I_{xx} + Ah^2$

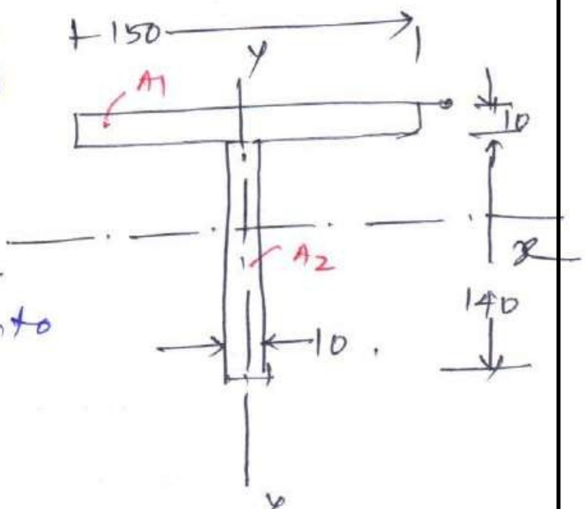
$$\Rightarrow \frac{\pi d^4}{128} = I_{xx} + \frac{\pi d^2}{8} \times \left(\frac{2d}{3\pi}\right)^2 =$$

$$\Rightarrow \frac{\pi d^4}{128} = I_{xx} + \frac{\pi d^2}{8} \times \frac{4d^2}{9\pi^2}$$

$$= I_{xx} + \frac{\pi d^4}{18\pi}$$

$$\Rightarrow \boxed{I_{xx} = \left(\frac{\pi d^4}{128} - \frac{d^4}{18\pi}\right)}$$

Moment of inertia of composite figures:-



Q.1 Determine the moment of inertia of the composite section about an axis passing through the centroidal axis. Also determine MI about axis of symmetry and radius of gyration.

Sol^n Dividing the composite area into A_1 and A_2

$A_1 = 150 \times 10 = 1500 \text{ mm}^2$

$A_2 = 140 \times 10 = 1400 \text{ mm}^2$

Distance of centroid from base of the composite figure

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{(A_1 + A_2)} = \frac{1500 \times 145 + 1400 \times 70}{2900}$$

$$= 108.79 \text{ mm}$$

Moment of inertia of the area about xx axis

$$I_{xx} = \left\{\frac{150 \times 10^3}{12} + 1500 \times (145 - 108.79)^2\right\} + \left\{\frac{10 \times 140^3}{12} + 1400 \times (108.79 - 70)^2\right\}$$

$$= (12500 + 1966746.15) + (2286666.667 + 2106529.74)$$

$$= 6372442.557 \text{ mm}^4$$

Similarly

$$I_{yy} = \frac{10 \times 150^3}{12} + \frac{140 \times 10^3}{12} = 2812500 + 11666.66667$$

$$= 2824166.667 \text{ mm}^4$$

Radius of gyration $k = \sqrt{\frac{I}{A}}$

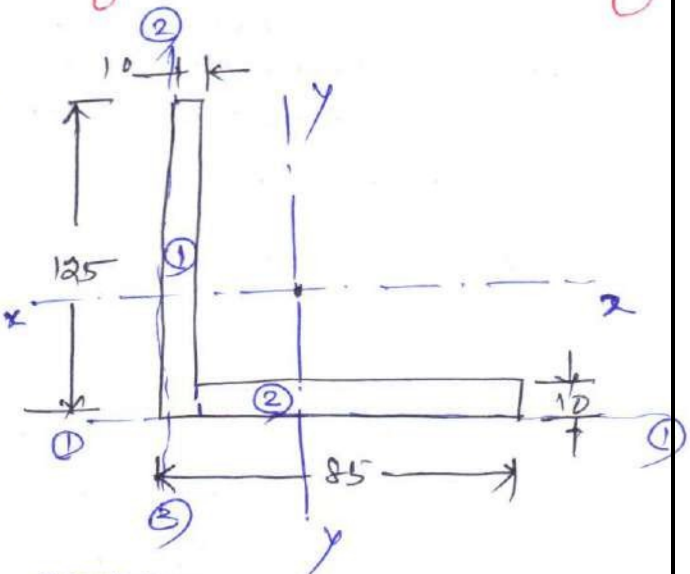
so $K_{xx} = \sqrt{\frac{I_{xx}}{A}}$

$= \sqrt{\frac{6372442.5}{2900}} = 46.87$ mm

Similarly $k_{yy} = \sqrt{\frac{I_{yy}}{A}} = \sqrt{\frac{2824166.667}{2900}}$

$= 31.206$ mm (Ans)

Q.2 Determine the MI of L-section about its centroidal axes parallel to the legs. Also find the polar moment of inertia.



We have $A_1 = 125 \times 10 = 1250$ mm^2

$A_2 = 75 \times 10 = 750$ mm^2

Total area $A_1 + A_2 = 2000$ mm^2

Distance of centroid from 1-1 axis

$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$

$= \frac{1250 \times 62.5 + 750 \times 5}{2000} = 40.9375$ mm

Distance of centroidal axis YY from 2-2 axis

$\bar{x} = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2}$

$= \frac{1250 \times 5 + 750 \times \left(\frac{75}{2} + 10\right)}{2000}$

$= \frac{1250 \times 5 + 750 \times 47.5}{2000} = 20.93$ mm

Moment of inertia about xx axis

$I_{xx} = \left\{\frac{10 \times 125^3}{12} + 1250 \times (62.5 - 40.9375)^2\right\}$

$+ \left\{\frac{75 \times 10^3}{12} + 750 \times (40.9375 - 5)^2\right\}$

$= (1627604.167 + 581176.7578) + (6250 + 968627.9297)$

$= \boxed{3183658.854 \text{ mm}^4}$

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Similarly MI about yy centroidal axis

$$I_{yy} = \left\{\frac{125\times10^3}{12} + 1250\times(20.93-5)^2\right\}$$

$$+ \left\{\frac{10\times75^3}{12} + 750\times(47.5-20.93)^2\right\}$$

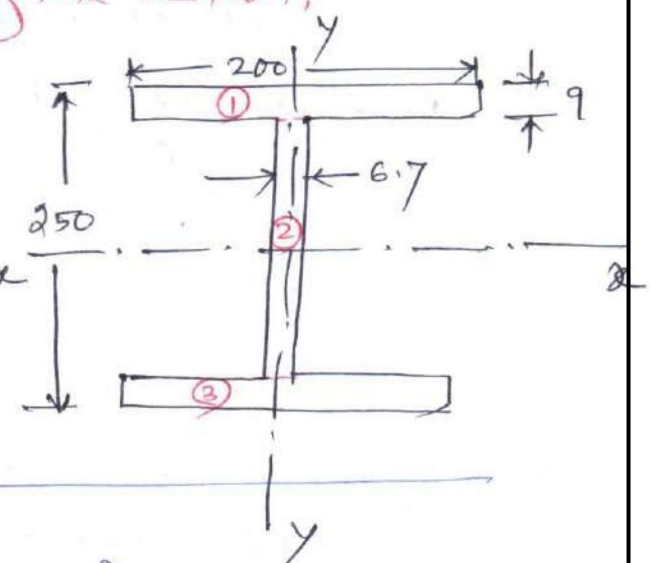
$$= (10416.66667 + 317206.125) + (351562.5 + 529473.675)$$

$$= \boxed{1208658.967\ mm^4}$$

Polar moment of inertia $I_{zz} = I_{xx} + I_{yy}$

$$= \boxed{4392317.821\ mm^4} \quad \text{(Ans)}$$

Q.3 Determine the MI of the symmetrical I section about its centroidal axes x-x and y-y. Also determine the polar moment of inertia of the section.



We have from the figure

$A_1 = 200\times9 = 1800\ mm^2$

$A_2 = 232\times6.7 = 1554.4\ mm^2$

$A_3 = 200\times9 = 1800\ mm^2$

~~Position of centroidal axis xx from base~~

$$\bar{Y} = \frac{A_1Y_1 + A_2Y_2 + A_3Y_3}{(A_1+A_2+A_3)}$$

$$= \frac{1800\times(4.5+232+9) + 1554.4\times(\frac{232}{2}+9) + 1800\times4.5}{(1800+1554.4+1800)}$$

$$= \frac{1800\times245.5 + 1554.4\times125 + 1800\times4.5}{(1800+1554.4+1800)}$$

$$= 125\ mm.$$

$$\bar{x} = \frac{A_1x_1 + A_2x_2 + A_3x_3}{(A_1+A_2+A_3)}$$

$$= \frac{1800\times100 + 1554.4\times96.65 + 1800\times100}{(1800+1554.4+1800)} = 98.98$$

MI about xx axis

$$I_{xx} = \left\{\frac{200\times 9^3}{12} + 1800\times(125-4.5)^2\right\} + \left\{\frac{6.7\times 232^3}{12} + 1554.4\times(\right.$$

$$+ \left\{\frac{200\times 9^3}{12} + 1800\times(125-4.5)^2\right\}$$

$$= (12150 + 26136450) + (6972002.133 + 0)$$

$$+ (12150 + 26136450)$$

$$= 26148600 + 6972002.133 + 26148600$$

$$= \boxed{59269202.13 \text{ mm}^4}$$

MI about yy axis

$$I_{yy} = \frac{9\times 200^3}{12} + \frac{232\times 6.7^3}{12} + \frac{9\times 200^3}{12}$$

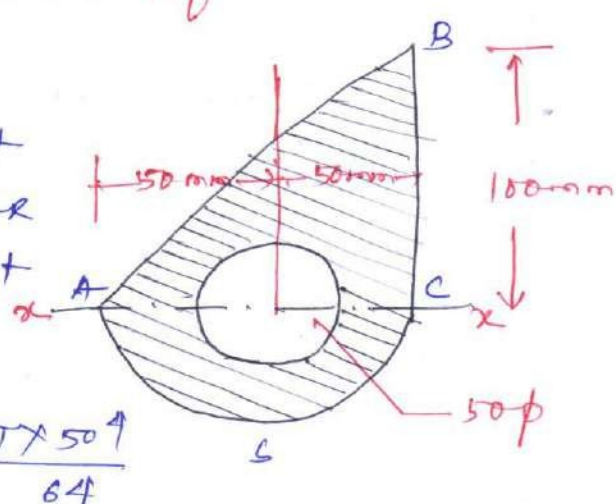
$$= 6000000 + 5814.751 + 6000000$$

$$= \boxed{12005814.75 \text{ mm}^4}$$

Polar moment of inertia $I_{zz} = I_{xx} + I_{yy}$

$$= \boxed{71275016.88 \text{ mm}^4}$$

Q.A Calculate the moment of inertia of the shaded area about xx axis.



MI of the shaded section about xx = MI of triangle ABC about xx + MI of semicircle ACS about xx − MI of circle

$$= \frac{100\times 100^3}{12} + \frac{\pi\times 100^4}{128} - \frac{\pi\times 50^4}{64}$$

$$= 8333333.333 + 2454369.261 - 306796.1576$$

$$= 10480906.44 \text{ mm}^4$$

$$= \boxed{1.048\times 10^7 \text{ mm}^4}$$

MODULE IV
PARTICLE DYNAMICS AND INTRODUCTION TO KINETICS

Rectilinear Translation :-

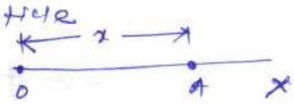
In statics, it was considered that the rigid bodies are at rest. In dynamics, it's considered that they are in motion, Dynamics is commonly divided into two branches. Kinematics and Kinetics.

In, kinematics we are concerned with space time relationship of a given motion of a body and not at all with the forces that cause the motion,

- In kinetics we are concerned with finding the kind of motion that a given body or system of bodies will have under the action of given forces or with what forces must be applied to produce a desired motion,

Displacement

from the fig. displacement of a particle can be defined by its x-coordinate, measured from the fixed reference point O.



- when the particle is to the right of fixed point O, this displacement can be considered positive and when it's towards the ~~rig~~ left hand side it's considered as negative.

General displacement time equation

$$x = f(t) \quad \text{--- (1)}$$

where $f(t)$ = function of time

for example $\boxed{x = c + bt}$

In the above equation c, represents the initial displacement at t = 0, while the constant b shows the rate at which displacement increases. It is called as uniform rectilinear motion.

Second example is $\boxed{x = \frac{1}{2} g t^2}$

where x is proportional to the square of time.

<u>Velocity</u>

<u>Acceleration</u>

<u>Example</u> The rectilinear motion of a particle is defined by the displacement - time equation $x = x_0 - v_0 t + \frac{1}{2} a t^2$. Construct displacement - time and velocity diagram for this motion and find the displacement and velocity at time $t_2 = 2s$. $x_0 = 750$ mm, $v_0 = 500$ mm/s, $a = 0.125$ m/s^2

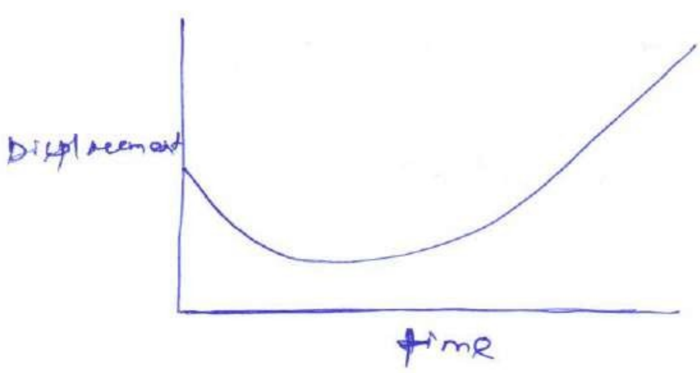
The equation of motion is

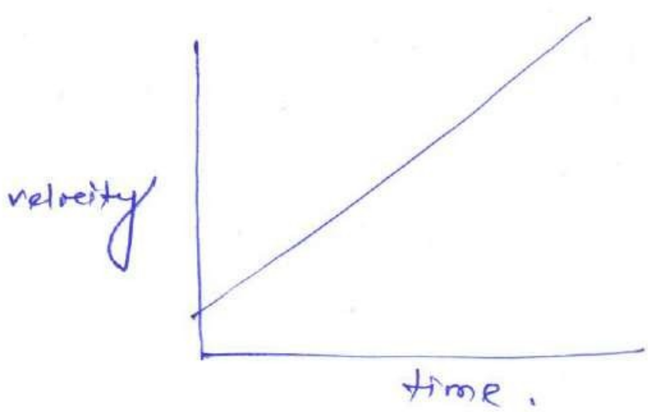
$$x = x_0 - v_0 t + \frac{1}{2} a t^2 \quad \text{---(1)}$$

$$v = \frac{dx}{dt} = -v_0 + at \quad \text{---(2)}$$

substituting x_0, v_0 and a in equation (1)

~~$x = 75 - 500$~~





Q-1 A bullet leaves the muzzle of a gun with velocity $v = 750$ m/s. Assuming constant acceleration from breech to muzzle find time t occupied by the bullet in travelling through gun barrel which is 750 mm long.

initial velocity of bullet $u = 0$
final velocity of bullet $v = 750$ m/s.
total distance $s = 0.75$ m.
$t = ?$

We have $v^2 - u^2 = 2as$,

$\Rightarrow v^2 = 2as \Rightarrow a = \frac{v^2}{2s} = \frac{750^2}{2 \times 0.75}$

$= 375000$ m/sec^2

Again $v = u + at$

$\Rightarrow 750 = 375000 \times t$

$\Rightarrow t = \frac{750}{375000} = \boxed{0.002 \text{ sec.}}$

Q-2 A stone is dropped into well and falls vertically with constant acceleration $g = 9.81$ m/sec^2. The sound of impact of stone on the bottom of well is heard after 6.5 sec. If velocity of sound is 336 m/s. How deep is the well?

$V = 336$ m/sec

let s = depth of well
t_1 = time taken by the stone into the well
t_2 = time taken by the sound to be heared.
total time $t = (t_1 + t_2) = 6.5$ sec.

Now $s = ut + \frac{1}{2} g t^2$

$\Rightarrow s = 0 + \frac{1}{2} g t^2$

$\Rightarrow t_1 = \sqrt{\frac{2s}{g}}$

When the sound travels with uniform velocity

$s = Vt_2$ or $t_2 = \frac{s}{V}$

$$\sqrt{\frac{2s}{g}} + \frac{s}{V} = 6.5$$

$$\Rightarrow \frac{2s}{g} = \left(6.5 - \frac{s}{336}\right)^2$$

$$\Rightarrow 2s = 9.81\left(6.5 - \frac{s}{336}\right)^2$$

$$= 9.81\left(\frac{2184 - s}{336}\right)^2$$

$$s = 0.0291\,(2184 - s)^2$$

$$= 0.0291\,(4769856 + s^2 - 4368s)$$

$$= 138802.809 + 0.0291 s^2 - 127.1055s$$

$$\Rightarrow 0.0291 s^2 - 129.1055s + 138802.809 = 0$$

$$\Rightarrow s =$$

$$0.2038s = 42.25 + 0.00000885 s^2 - 0.0386s$$

~~s = 174~~

$$0.00000885 s^2 - 0.1658s + 42.25 = 0$$

$$\boxed{s = 17.31\,m.}$$

A2 A rope AB is attached at B to a small block of negligible dimensions and passes over a pulley so that it's free end A hangs 1.5 m above C ground when the block rests on the floor. The end A of the rope is moved horizontally in a str. line by a man walking with a uniform velocity $v_0 = 3\,m/s$. Plot the velocity-time diagram

(b) find the time t required for the block to reach the pulley if h = 4.5 m, pulley dimensions are negligible.

A3 A particle starts from rest and moves along a str. line with constant acceleration a. If it acquires a velocity $v = 3\,m/s$. after having travelled a distance s = 7.5 m, find magnitude of acceleration.

Principles of Dynamics:

Newton's law of motion!

First law: Everybody continues in it's state of rest or of uniform motion in a straight line except in so far as it may be compelled by force to change that state.

Second Law:-

The acceleration of a given particle is proportional to the force applied to it and takes place in the direction of the straight line in which the force acts.

Third law To every action there is always an equal and contrary reaction or the mutual actions of any two bodies are always equal and oppositely directed.

General Equation of Motion of a Particle!

$$\boxed{ma = f}$$

Differential equation of Rectilinear motion!.

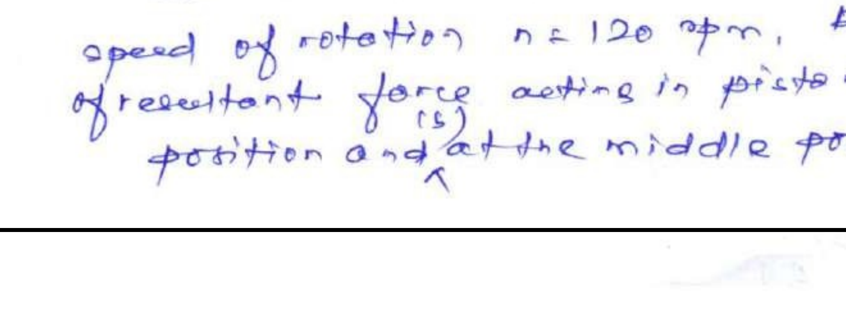
Differential form of equation for rectilinear motion can be expressed as

$$\boxed{\frac{W}{g}\ddot{x} = X}$$

where $\ddot{x}$ = acceleration

X = Resultant acting force.

Example



For the engine shown in fig., the combined wt. of piston and piston rod W = 450N., crank radius r = 250mm and uniform speed of rotation n = 120 rpm. Determine the magnitude of resultant force acting in piston (a) at extreme position and (b) at the middle position piston has a simple harmonic motion represented by displacement-time equation

$$x = r\cos\omega t \quad \text{---(1)}$$

$$\omega = \frac{2\pi n}{60} = \frac{2\pi \times 120}{60} = 4\pi \text{ rad/s.}$$

$$\dot{x} = -r\omega \sin\omega t$$

$$\ddot{x} = -r\omega^2 \cos\omega t \quad \text{---(2)}$$

Differential equation of motion

$$\frac{W}{g}\ddot{x} = X$$

$$\Rightarrow -\frac{W}{g} r\omega^2 \cos\omega t = X$$

$$\Rightarrow X = -\frac{450}{9.81} \times 0.25 \,(4\pi)^2 \cos(4\pi t)$$

for extreme position

$\cos\omega t = -1$

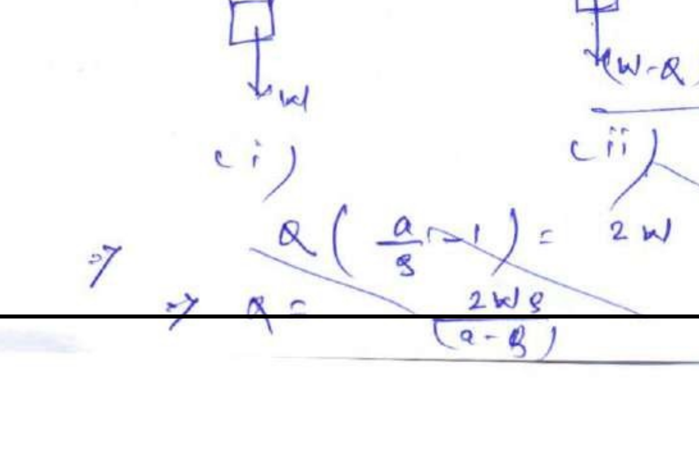
so $\boxed{X = 1810N.}$

~~for extre~~ middle position $\cos\omega t = 0$.

so Resultant force $= 0$.

E-2 A ballon of gross wt W is falling vertically downward with constant acceleration a. What amount of ballast Q must be thrown out in order to give ballon an equal upward acceleration a.

P = buoyant force.



(i) considering 1st case when ballon is falling,

$$\frac{W}{g} a = W - P \quad \text{---(1)}$$

(ii) $$\frac{W-Q}{g} a = P - (W-Q) \quad \text{---(2)}$$

Eq(1) + Eq(2)

$$\frac{Q}{g} \cdot a = W + W - Q = 2W + Q$$

$$\Rightarrow Q\left(\frac{a}{g} - 1\right) = 2W$$

$$\Rightarrow Q = \frac{2Wg}{(a-g)}$$

$$\frac{Wa}{g} = (W-P)$$

$$\frac{(W-Q)a}{g} = P-(W-Q)$$

$$\frac{Wa+(W-Q)a}{g} = W-\cancel{P}+\cancel{P}-(W-Q) = Q$$

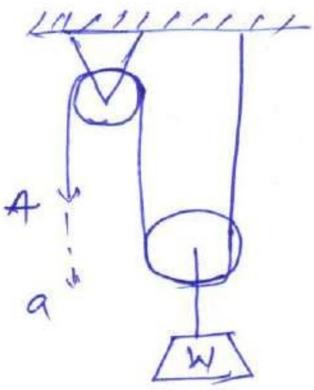
$$\Rightarrow \frac{Wa+Wa-Qa}{g} = Q$$

$$\Rightarrow 2Wa = Qg+Qa$$

$$\Rightarrow \boxed{Q = \frac{2Wa}{(g+a)}}$$

Q.1 A wt. W = 4450N is supported in a vertical plane by string and pulleys arranged shown in fig. If the free end A of ~~the~~ the string is pulled vertically downward with constant acceleration a = 18 m/s2 find tension S in the string.

Differential equation of motion for the system is



$$2S-W = \frac{W}{g}\times\frac{a}{2}$$

$$\Rightarrow 2S = W+\frac{Wa}{2g}$$

$$= \cancel{\frac{W}{2}\left(2+\frac{a}{2g}\right)}$$

$$= W\left(1+\frac{a}{2g}\right)$$

$$\Rightarrow S = \frac{W}{2}\left(1+\frac{a}{2g}\right)$$

$$= \frac{4450}{2}\left(1+\frac{18}{2\times9.81}\right) = \boxed{4266.28\ N.}$$

$$\frac{Wa}{g} = (W - P)$$

$$\frac{(W-Q)a}{g} = P - (W - Q)$$

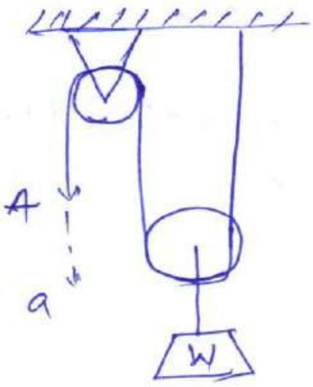
$$\frac{Wa + (W-Q)a}{g} = W - \cancel{P} + \cancel{P} - (W - Q) = Q$$

$$\Rightarrow \frac{Wa + Wa - Qa}{g} = Q$$

$$\Rightarrow 2Wa = Qg + Qa$$

$$\Rightarrow \boxed{Q = \frac{2Wa}{(g+a)}}$$

Q.1 A wt. $W = 4450N$ is supported in a vertical plane by string and pulleys arranged shown in fig. If the free end A of ~~oth~~ the string is pulled vertically downward with constant acceleration $a = 18\,m/s^2$ find tension S in the string.



Differential equation of motion for the system is

$$2S - W = \frac{W}{g} \times \frac{a}{2}$$

$$\Rightarrow 2S = W + \frac{Wa}{2g}$$

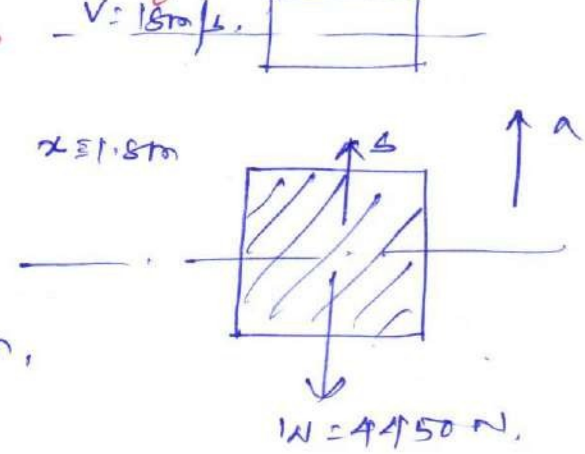
$$= \cancel{\frac{W}{2}\left(2 + \frac{a}{2g}\right)}$$

$$= W\left(1 + \frac{a}{2g}\right)$$

$$\Rightarrow S = \frac{W}{2}\left(1 + \frac{a}{2g}\right)$$

$$= \frac{4450}{2}\left(1 + \frac{18}{2 \times 9.81}\right) = \boxed{4266.28\ N.}$$

Q. 2 An elevator of gross wt $W = 4450N$ starts to move upward direction with a constant acceleration and acquires a velocity $v = 18 m/s$, after travelling a distance $= 1.8 m$, find tensile force S in the cable during it's motion.



$W = 4450 N.$

$V = 18 m/s.$

initial velocity $u = 0$

distance travelled $x = 1.8 m,$

$$S - W = \frac{W}{g} \cdot a$$

$$\Rightarrow S = W + \frac{W}{g} a = W\left(1 + \frac{a}{g}\right) \quad \text{--- (1)}$$

Now applying equation of kinematics

$$v^2 - u^2 = 2as$$

$$\Rightarrow 18^2 - 0 = 2a \times 1.8$$

$$\Rightarrow a = \frac{18^2}{2 \times 1.8} = \boxed{90 \, m/s^2}$$

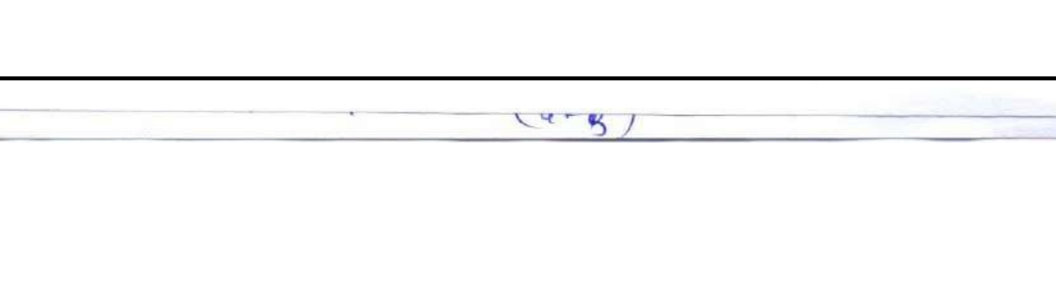
substituting the values of a in eq. (1)

$$S = 4450\left(1 + \frac{90}{9.81}\right) = \boxed{45275.7 \, N.}$$

(A-1)

Q. 3 A train weighing 1870N without the locomotive starts to move with constant acceleration along a straight track and in first 60s acquires a velocity of 56 Kmph. Determine the tension S in draw bar both locomotive and train if the air resistance is 0.005 times the wt. of the train.

$V = 56$ Kmph $= 15.56 m/s.$



$S - F = \frac{W}{g} \cdot a$

$\Rightarrow S = 0.005W + \frac{Wa}{g}$ —(1)

from eq. of kinematics.

$v = u + at$

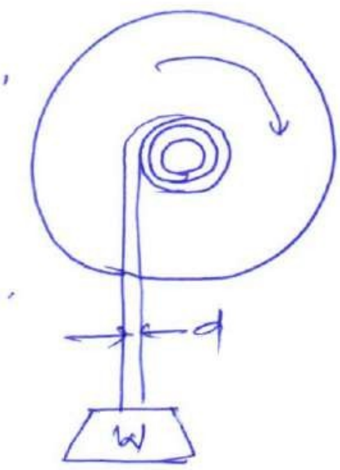
$\Rightarrow a = \left(\frac{15.56 - 0}{60}\right) = 0.26\ m/sec^2$

substituting the value of a in eq. (1)

$S = W\left(0.005 + \frac{a}{g}\right)$

$= 1870\left(0.005 + \frac{0.26}{9.81}\right) = \boxed{58.9\ kN.}$

A-2 Q.4 A wt. W is attached to the end of a small flexible rope of dia. d = 6.25mm. and is raised vertically by winding the rope on a reel. if the reel is turned uniformly at a rate of 2 rps. what will be the tension in rope.



dia of rope d = 6.25mm = 0.00625m.

No of revolutions N = 2rps.

let x = initial radius of reel.

t = time taken for N revolutions.

Net radius after t sec.

$R = [x + (Nt d)]$

Now mass velocity $v = R\omega$

$\omega = 2\pi N.$

$\therefore v = (x + Ntd)\ 2\pi N$

acceleration of rope = $a = \frac{dv}{dt}$

$a = \frac{d}{dt}\left[2\pi N x + 2\pi N^2 t d\right] = 2\pi N^2 d$

$S - W = \frac{W}{g} \cdot a \Rightarrow S = W + \frac{Wa}{g} = W\left(1 + \frac{a}{g}\right)$

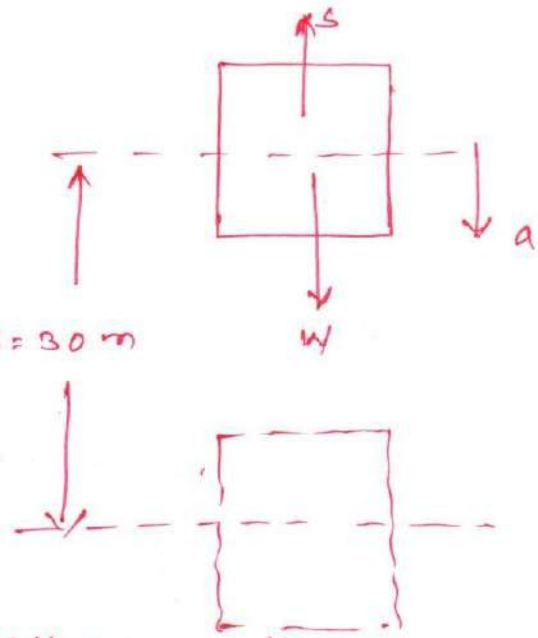
$\Rightarrow S = W\left(1 + \frac{2\pi N^2 d}{g}\right)$ ~~$= W(1+$~~

$$\Rightarrow S = W\left(1 + \frac{2\pi \times 2^2 \times 0.00625}{9.81}\right)$$

$=$

Ans-3

Q.5 A mine cage of wt $W = 8.9$ kN starts from rest and moves downward with constant acceleration travelling a distance $s = 30$ m in 10 sec. Find the tensile force in the cable.



Wt. of cage $W = 8.9$ kN.
initial velocity $u = 0$.
distance travelled $s = 30$ m
time $t = 10$ sec.

$$s = \cancel{ut}^{0} + \frac{1}{2}at^2$$

$$\Rightarrow 30 = \frac{1}{2}a \times 10^2$$

$$\Rightarrow t = \frac{60}{10^2} = \boxed{0.6 \text{ m/sec}^2}$$

Differential equation of rectilinear motion

$$W - S = \frac{W}{g} \cdot a$$

$$\Rightarrow S = W - \frac{W}{g}a = W\left(1 - \frac{a}{g}\right)$$

$$= 8.9\left(1 - \frac{0.6}{9.81}\right)$$

$$\Rightarrow \boxed{S = 8.35 \text{ kN.}}$$ (Ans)

D'Alembert's Principle

Differential equation of motion (rectilinear) can be written as

$$X - m\ddot{x} = 0 \quad \text{---} (1)$$

Where X = Resultant of all applied force in the direction of motion

m = mass of the particle

The above equation may be treated as equation of dynamic equilibrium. To express this equation, in addition to the real force acting on the particle a fictitious force $m\ddot{x}$ is required to be considered. This force is equal to the product of mass of the particle and it's acceleration and directed in opposit direction, and is called the inertia force of the particle.

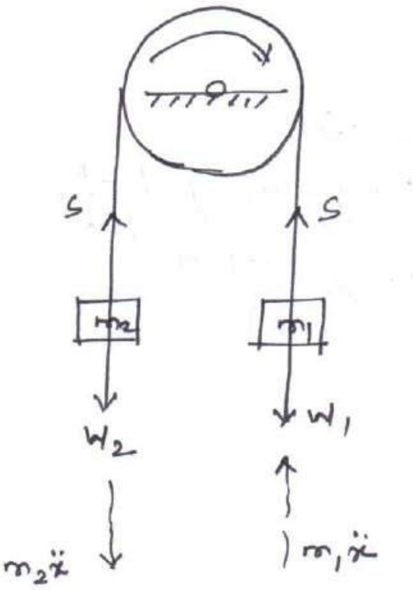
$$\boxed{-\Sigma m\ddot{x} = -\ddot{x}\,\Sigma m = -\frac{W}{g}\ddot{x}}$$

Where W = total weight of the body

so the equation of dynamic equilibrium can be expressed as:

$$\boxed{\Sigma X_i + \left(-\frac{W}{g}\ddot{x}\right) = 0} \quad \text{---} (2)$$

Example 1



for the example shown considering the motion of pulley as shown by the arrow mark. where upward acceleration $\ddot{x}_2$ for W_2 and downward acceleration $\ddot{x}_1$ for W_1

- corresponding inertia forces and their direction are indicated by dotted line.
- By adding inertia forces to the real forces (such as W_1, W_2 and tension in strings) we obtain, for each particle, a system of forces in equilibrium.

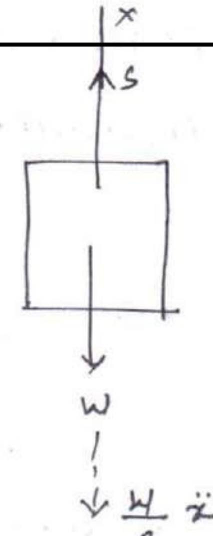
The equilibrium equation for the entire system without S

$$W_2 + m_2\ddot{x} = W_1 - m_1\ddot{x}$$

$$\Rightarrow (m_1 + m_2)\ddot{x} = (W_1 - W_2) \Rightarrow \boxed{\ddot{x} = \frac{W_1 - W_2}{(W_1 + W_2)} \cdot g}$$

Example 2 A body is moving in upward direction by a rope.

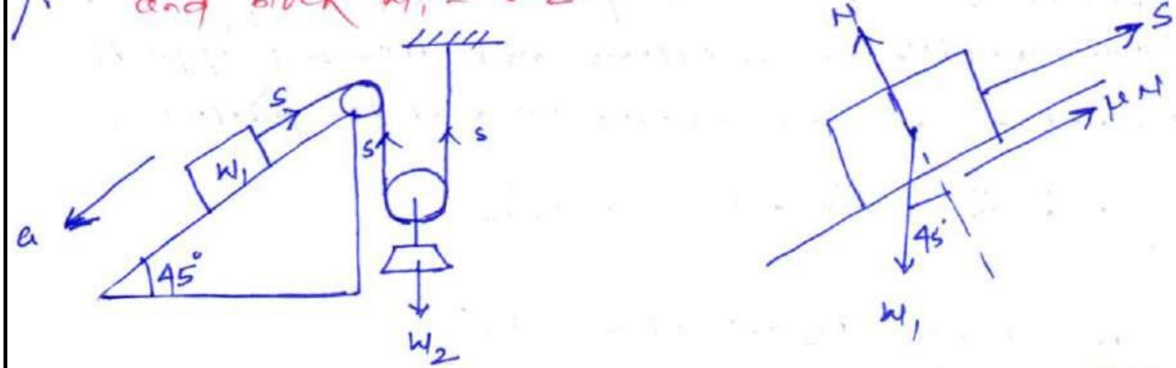
So the equation of dynamic equilibrium considering the real and inertia forces.



$S - W - \frac{W}{g} a = 0$, so tensile force in rope

$\Rightarrow \boxed{S = W\left(1 + \frac{a}{g}\right)}$

Q.1 Find tensions in the string during motion of the system ~~(a)~~ if $W_1 = 900N$, $W_2 = 450N$. ~~The~~ μ betn the inclined plane and block $W_1 = 0.2$



When W_1 moves downward in the inclined plane with an ~~ac~~ acceleration a, then acceleration of $W_2 = \frac{a}{2}$

Considering dynamic equilibrium of W_1, from D'Alembert's principle

$$\left(W_1 \sin 45^\circ - \mu N - S\right) - \frac{W_1}{g} a = 0$$

$$\Rightarrow \frac{W_1}{g} a = W_1 \sin 45^\circ - \mu N - S$$

$$= W_1 \sin 45 - \mu W_1 \cos 45 - S$$

$$\Rightarrow a = \left(900 \times \frac{1}{\sqrt{2}} - 0.2 \times 900 \times \frac{1}{\sqrt{2}} - S\right)\frac{9.81}{900}$$

$$= (636.4 - 127.28 - S)\ 0.0109$$

$$\Rightarrow a = \cancel{5.549}\ 6.93676 - 1.387352 - 0.0109 S \quad \text{--- (1)}$$

Similarly ~~from~~ for weight W_2

$$2S - W_2 - \frac{W_2}{g}\frac{a}{2} = 0$$

$$\Rightarrow \cancel{\frac{W_2 a}{2g}} = W_2\left(1 + \frac{a}{2g}\right) = 2S$$

$$\Rightarrow \cancel{2}S = \frac{450}{2}\left(1 + \frac{a}{19.62}\right) = 225 + 11.46\,a \quad \text{--- (2)}$$

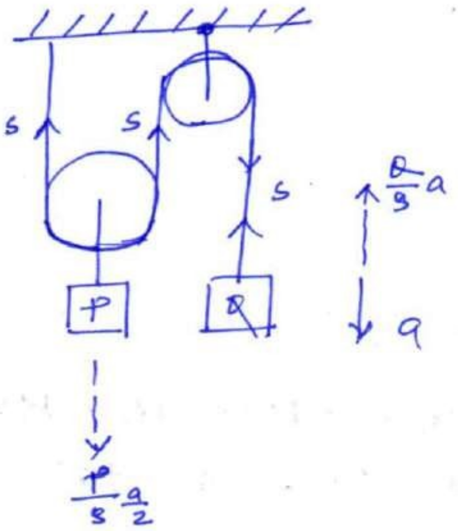
Substituting the value of S in eq. (1)

~~$a = 5.549 - 2.4525 - 0.125a$~~ $\Rightarrow a =$

$$a = 6.93676 - 1.387352 - 0.0109\,(225 + 11.46\,a)$$
$$= \cancel{6.93}\;5.549408 - 2.4525 - 0.124914\,a$$
$$= 3.096908 - 0.124914\,a$$
$$\Rightarrow \boxed{a = 2.75\ m/s^2}$$

Q.2 Two weights P and Q are connected by the arrangement shown in fig. Neglecting friction and inertia of pulley and cord find the acceleration a of wt. Q

Assume $P = 178\ N$, $Q = 133.5\ N$.



Applying D'Alembert's principle fo. Q

$$Q - S - \frac{Q}{g}a = 0$$
$$\Rightarrow S = Q\left(1 - \frac{a}{g}\right) \quad -(1)$$
$$= 133.5\left(1 - \frac{a}{9.81}\right)$$

Applying D'Alembert's principle to P

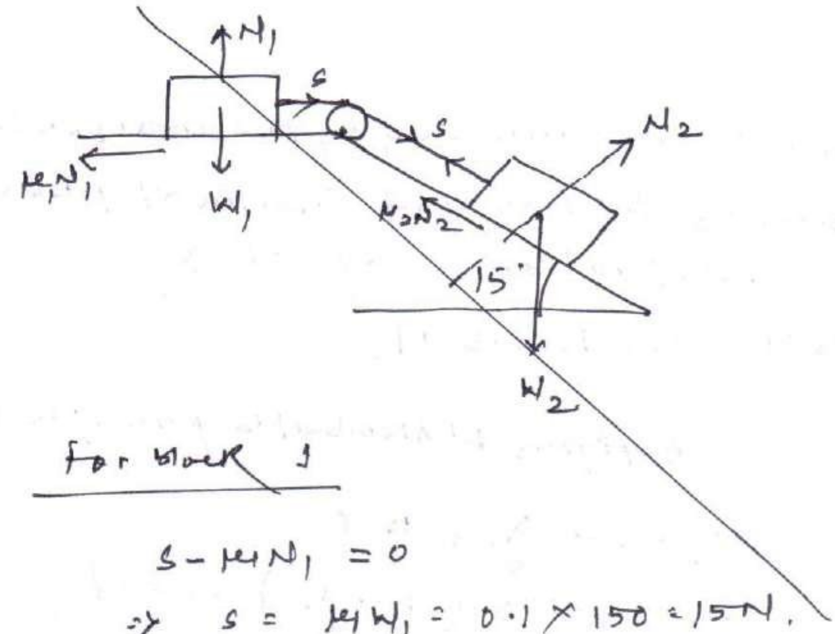
$$2S - P - \frac{Pa}{2g} = 0$$
$$\Rightarrow 2S = P\left(1 + \frac{a}{2g}\right)$$
$$\Rightarrow S = \frac{P}{2}\left(1 + \frac{a}{2g}\right) \quad -(2)$$
$$= \frac{178}{2}\left(1 + \frac{a}{19.62}\right)$$

$$133.5\left(1 - \frac{a}{9.81}\right) = 89\left(1 + \frac{a}{19.62}\right)$$
$$\Rightarrow 133.5 - 13.608\,a = 89 + 4.536\,a$$
$$\Rightarrow 18.144\,a = 44.5$$
$$\Rightarrow \boxed{a = 2.45\ m/s^2} \quad (\text{Ans})$$

~~**Q.3** Assuming the car in the fig. to have a velocity of 6 m/s find shortest distance s in which it can be stopped with constant deceleration without disturbing the block. Data: c = 0.6 m, h = 0.9 m~~

$\mu = 0.5$

Q.3 Two blocks of wt $W_1 = 150N$ and $W_2 = 500N$ are connected by an inextensible string. Find the acceleration of the blocks and tension in the string. $\mu_1 = 0.1$, $\mu_2 = 0.$



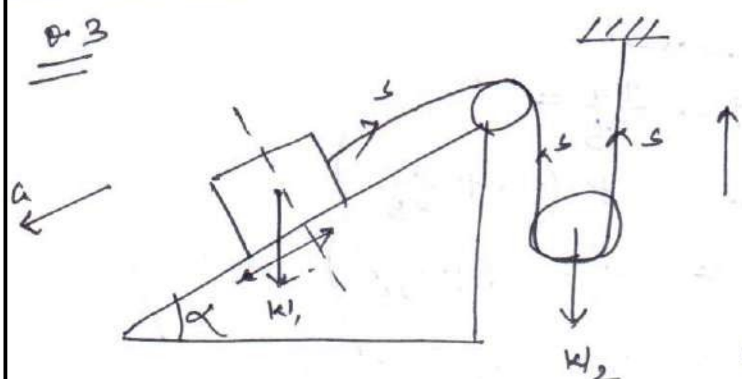
for block 1

$S - \mu_1 N_1 = 0$

$\Rightarrow S = \mu_1 W_1 = 0.1 \times 150 = 15N.$

for block 2

Q.3



$W_1 = 890N \quad W_2 = 445N.$

$\mu = 0.2 \quad \alpha = 45°$

find S.

considering equilibrium of W_1 and applying D'Alembert's principle

$$W_1 \sin 45° - \mu N_1 - S - \frac{W_1}{g} a = 0$$

$$\Rightarrow S = W_1 \sin 45 - \mu N - \frac{W_1}{g} a$$

$$= \frac{890}{\sqrt{2}} - 0.2 \times 890 \times \frac{1}{\sqrt{2}} - \frac{890}{9.81} a$$

$$= 629.32 - 125.865 - 90.72a$$

$$\boxed{S = 503.455 - 90.72a} \quad \text{—(1)}$$

Applying D'Alembert's principle for W_2

$$2S - W_2 - \frac{W_2}{g}\frac{a}{2} = 0$$

$$\Rightarrow 2S = W_2\left(1 + \frac{a}{2g}\right)$$

$$\Rightarrow S = \frac{W_2}{2}\left(1 + \frac{a}{2g}\right) = \frac{445}{2}\left(1 + \frac{a}{19.62}\right) = 222.5 + 11.34a \quad \text{—(2)}$$

equating (1) and (2)

$503.455 - 90.72a = 222.5 + 11.34a$

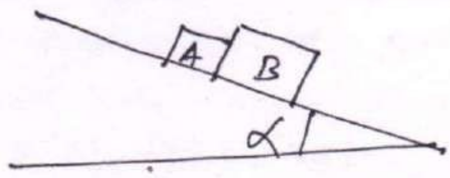
$\Rightarrow 102.0604\,a = 280.955$

$\Rightarrow \boxed{a = 2.75\ m/s^2}$

so $S = 222.5 + 11.34 \times 2.75$

$= \boxed{253.71\ N.}$

Q.4

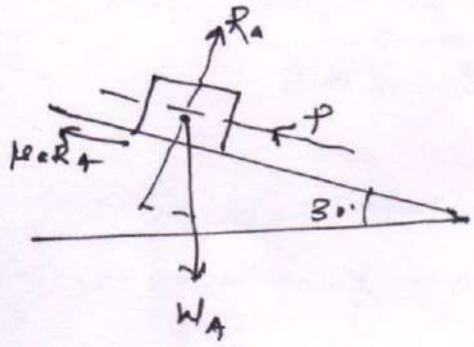


$W_A = 44.5N \quad W_B = 89N$

$\alpha = 30° \quad \mu_a = 0.15$

$\mu_B = 0.3$

find pressure P betn blocks.



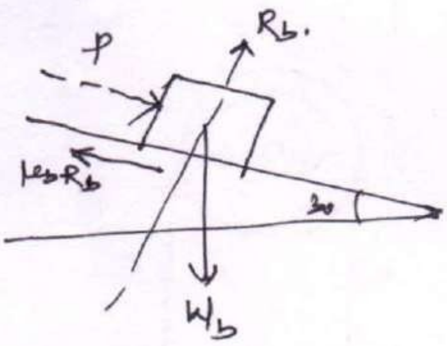
$W_a \sin 30 - P - \mu_a R_a - \frac{W_a}{g} a = 0$

$\Rightarrow P = W_a \sin 30 - \mu_a R_a - \frac{W_a}{g} a$

$= 44.5 \times \frac{1}{2} - 0.15 \times 44.5 \times \cos 30 - \frac{44.5}{9.81} a$

$= 22.25 - 5.78 - 4.53a$

$= 16.47 - 4.53a \quad \text{---(1)}$



$P + W_b \sin 30 - \mu_b R_b - \frac{W_b}{g} a = 0$

$\Rightarrow P = -\frac{W_b}{2} + 0.3 \times 89 \cos 30 + \frac{89}{9.81} a$

$= -\frac{89}{2} + 23.122 + 9.07a$

$= -21.378 + 9.07a \quad \text{---(2)}$

$16.47 - 4.53a = -21.378 + 9.07a$

$\Rightarrow 13.6a = 37.848$

$\Rightarrow a = 2.78\ m/s^2$

$P = 3.87\ N.$

Momentum and Impulse

We have the differential equation of rectilinear motion of a particle

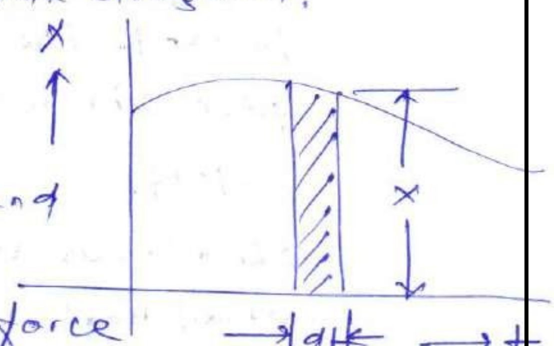
$$\frac{W}{g}\ddot{x} = X$$

Above equation may be written as

$$\frac{W}{g}\frac{d\dot{x}}{dt} = X$$

or $$\boxed{d\left(\frac{W}{g}\dot{x}\right) = X\,dt} \quad \text{---(1)}$$

In the above equation we will assume force X as a function of time represented by a force time diagram.



The right hand side of eq. (1) is then represented by the area of shaded elemental strip of ht X and width dt. This quantity i.e ($X\,dt$) is called impulse of the force X in time dt. The expression on the left hand side of the expression $\left(\frac{W}{g}\dot{x}\right)$ is called momentum of particle.

So the eq. (1) represents the differential change in momentum of a particle in time dt.

Integrating eq. (1) we have

$$\boxed{\frac{W}{g}\dot{x} + C = \int_0^t X\,dt} \quad \text{---(2)}$$

where C is a constant of integration

Now assuming an initial moment, $t=0$, the particle has an initial velocity $\dot{x}_0$

so $$\boxed{C = -\frac{W}{g}\dot{x}_0} \quad \text{---(3)}$$

So equation (2) becomes

$$\boxed{\frac{W}{g}\dot{x} - \frac{W}{g}\dot{x}_0 = \int_0^t X\,dt} \quad \text{---(4)}$$

from equation (4) it is clear that the total change of momentum of a particle during a finite interval of time is equal to the impulse of acting force.

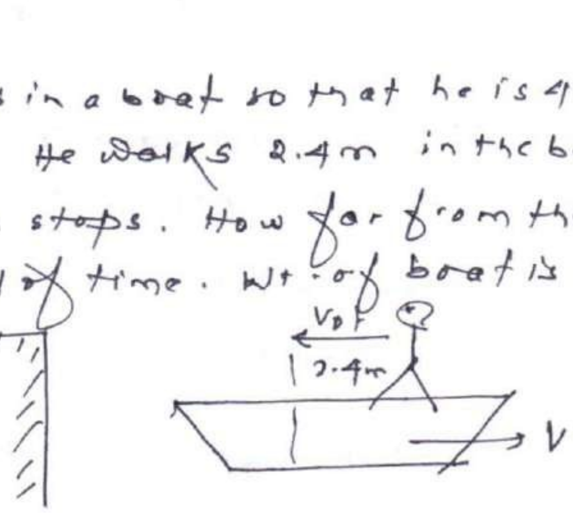
in other words

$$\boxed{F \cdot dt = d(mv)}$$

where $m \times v =$ momentum

~~Recoil of gun~~

Q-1 A man of wt 712 N stands in a boat so that he is 4.5 m from a pier on the shore. He walks 2.4 m in the boat towards the pier and then stops. How far from the pier will he be at the end of time. Wt of boat is 890 N.



wt of man $W_1 = 712\,N$

wt of boat $W_2 = 890\,N$

Let v_0 is the initial velocity of man and t is time

then $v_0 t = x$

$\Rightarrow v_0 t = 2.4\,m$

$\Rightarrow v_0 = \left(\frac{2.4}{t}\right)$ m/s.

let V = velocity of boat towards right

according to conservation of momentum

$$W_1 v_0 = (W_1 + W_2) V$$

$$\Rightarrow V = \frac{W_1 v_0}{(W_1 + W_2)}$$

distance covered by boat

$$s = V \cdot t = \frac{W_1 v_0}{(W_1 + W_2)} \cdot t$$

$$\Rightarrow s = \frac{712 \times 2.4}{\cancel{t}(712 + 890)} \cdot \cancel{t} = \boxed{1.067\,m}$$

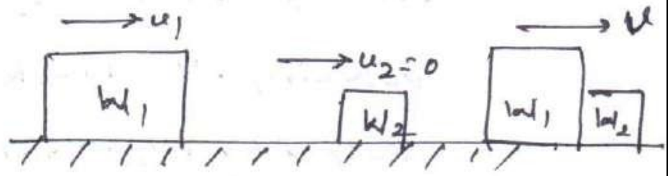
position of man from pier

$= 4.5 + s - x$

$= 4.5 + 1.667 - 2.4 = \boxed{3.167\,m}$ (Ans).

Q.2 A locomotive wt 534 kN has a velocity of 16 kmph and backs into a frieght car of wt 86 kN that is at rest on a track. after coupling at what velocity v the entire system continues to move. Neglect friction.

conservation of momentume



$W_1 u_1 + W_2 u_2 = (W_1 + W_2) V$

$\Rightarrow V = \dfrac{534 \times 4.45}{(534 + 86)} = \boxed{3.82\ m/s.}$

Q.3 A 667.5 man sits in a 333.75 N canoe and fire a rifle bullet horizontally. ~~directed over~~ find velocity v with which the canoe will move after the shot. the rifle has a muzzle velocity 660 m/s and wt. of bullet is 0.28 N.

Wt. of man W_1 = 667.5 N.

Wt. of canoe W_2 = 333.75 N.

Wt. of bullet W_2 = 0.28 N.

velocity of muzzle u = 660 m/s.

V = final velocity of canoe.

According to conservation of momentum

$\cancel{W_3 u}\ W_3 u = (W_1 + W_2) V$

$\Rightarrow V = \dfrac{0.28 \times 660}{(667.5 + 333.75)} = \boxed{0.182\ m/s.}$

Q.4 A wood block wt 22.25 N rests on a smooth horizontal surface. A revolver bullet weighing 0.14 N is shot horizontally into the side of block. If the block attains a velocity of 3 m/s what is muzzle velocity.

Wt. of wood block $W_1 = 22.25\,N$.

Wt. of bullet $W_2 = 0.14\,N$.

velocity of block $V = 3\,m/s$.

velocity of muzzle $= u$

According to conservation of momentum

$$\cancel{W_1 v} = W_2 u = (W_1 + W_2) V$$

$$\Rightarrow u = \frac{(22.25 + 0.14)\,3}{0.14}$$

$$= \boxed{479.98\ m/s.}$$

Conservation of momentum

When the sum of impulses due to external force is zero the momentum of the system remain conserved

When $\sum \int_0^t X\,dt = 0$

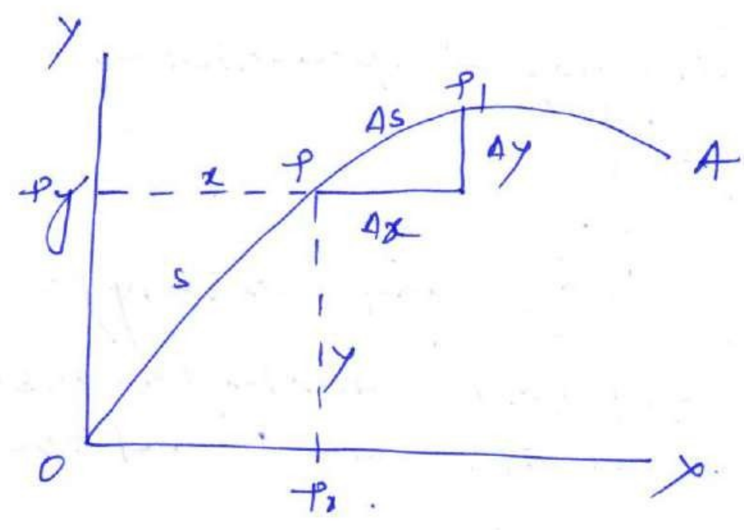
$$\boxed{\sum \left(\frac{W}{g}\right) \dot{x}_2 = \sum \left(\frac{W}{g}\right) \dot{x}_1}$$

$\therefore$ final momentum = initial momentum.

Curvilinear Translation

When a moving particle describes a curved path it is said to have curvilinear motion.

Displacement



consider a particle P in a plane on a curved path. To define the particle we need two coordinates x and y. as the particle moves, these coordinates ~~move~~ change with time and the displacement time equations are

$$x = f_1(t) \qquad y = f_2(t) \qquad -(1)$$

The motion of particle can also be expressed as

$$y = f(x) \qquad s = f_1(t)$$

where $y = f(x)$ represents the equation of path OA and $s = f_1(t)$ gives displacement s measured along the path as a function of time.

velocity :-

Considering an infinitesimal time difference from t to $t + \Delta t$ during which the particle moves from P to P_1 along it's path.

then velocity of particle may be expressed as

$$\boxed{\bar{v}_{av} = \frac{\Delta s}{\Delta t}}$$

$$\boxed{\begin{aligned}(v_{av})_x &= \frac{\Delta x}{\Delta t} \\ (v_{av})_y &= \frac{\Delta y}{\Delta t}\end{aligned}}$$

(average velocity along x and y coordinates)

It can also be expressed as

$$v_x = \frac{dx}{dt} = \dot{x}$$

$$v_y = \frac{dy}{dt} = \dot{y}$$

so the total velocity may be represented by

$$\boxed{v = \sqrt{\dot{x}^2 + \dot{y}^2}}$$

and $\cos(v, x) = \frac{\dot{x}}{v}$ and $\cos(v, y) = \frac{\dot{y}}{v}$

where (v, x) and (v, y) denotes the angles betn the direction of velocity vector $\overline{v}$ and the coordinate axes.

Acceleration :-

The acceleration particles may be described as

$$\boxed{\begin{aligned} a_x &= \frac{d\dot{x}}{dt} = \ddot{x} \\ a_y &= \frac{d\dot{y}}{dt} = \ddot{y} \end{aligned}}$$

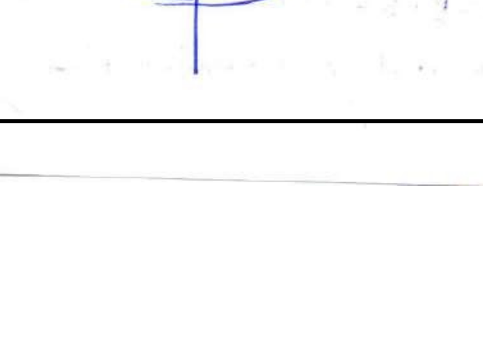
It is also known as instantaneous acceleration

Total acceleration $a = \sqrt{\ddot{x}^2 + \ddot{y}^2}$

Considering particular path for above case.

$x = r\cos\omega t \quad y = r\sin\omega t.$

$x^2 + y^2 = r^2$



$\dot{x} = -r\omega\sin\omega t \quad \dot{y} = r\omega\cos\omega t$

$v = \sqrt{\dot{x}^2 + \dot{y}^2}$

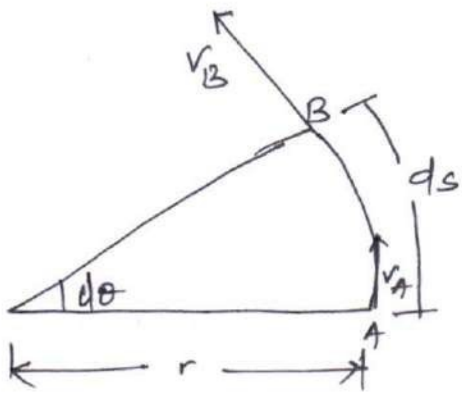
$\ddot{x} = -r\omega^2\cos\omega t \quad \ddot{y} = -r\omega^2\sin\omega t$

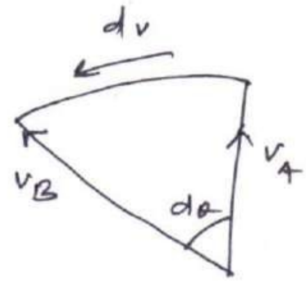
$a = \sqrt{\ddot{x}^2 + \ddot{y}^2}$

D'Alembert's Principle in Curvilinear Motion

① ②

Acceleration during circular motion





V_A = tangential velocity at A
= tangential velocity at B
$= V_B = V$

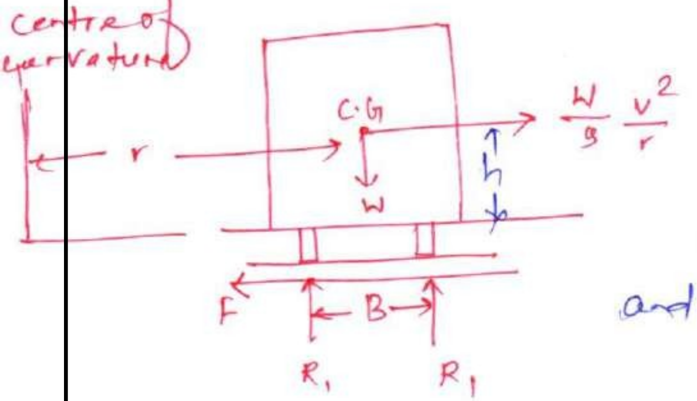
Now $dv = v\,d\theta = v\dfrac{ds}{r} = \dfrac{v}{r}ds$

acceleration $= \dfrac{dv}{dt} = \boxed{\dfrac{v^2}{r}}$

so when a body moves with uniform velocity v along a curved path of radius r, it has a radial inward acceleration of magnitude $\dfrac{v^2}{r}$

Applying D'Alembert's principle to get equilibrium condition an inertia force of magnitude $\dfrac{W}{g}a$
$= \dfrac{W}{g}\dfrac{v^2}{r}$ must be applied in outward direction
it is known as centrifugal force.

Motion on a level road



Consider a body is moving with uniform velocity on a curvilinear curve of radius r. Let the road is flat.
Let W = wt. of the body
and inertia force is given by

$$\frac{W}{g}a = \frac{W}{g}\frac{v^2}{r}$$

Condition for skidding :-

Let W = wt. of vehicle

R_1, R_2 = reactions at wheel

F = frictional force.

$\frac{W}{g} \cdot \frac{v^2}{r}$ = inertia force

Skidding takes place when the frictional forces reaches limiting value i.e

$F = \mu W$

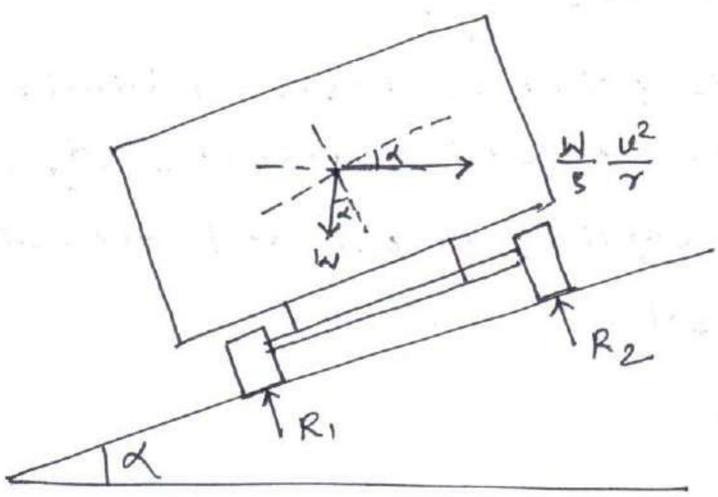
Then maxm permissible speed to avoid skidding

$$\boxed{v = \sqrt{\frac{gr}{2} \frac{B}{h}}}$$

The distance betn inner and outer wheel is equal to the gauge of railway track and expressed as G.

so $$\boxed{v = \sqrt{\frac{gr}{2} \frac{G}{h}}}$$

Designed speed and angle of Braking



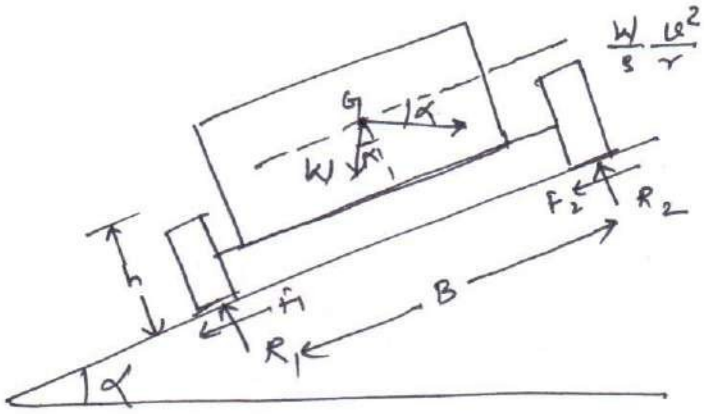
Σ of all the forces in the inclined plane

$\frac{W}{g} \frac{v^2}{r} \cos\alpha - W \sin\alpha = 0$

$\Rightarrow \boxed{\tan\alpha = \frac{v^2}{gr}}$

Relation betn the angle of braking and designed speed is $\boxed{\tan\alpha = \frac{v^2}{gr}}$

condition for skidding and overturning:—



(a) condition for skidding

$$\boxed{v = \sqrt{\tan(\alpha + \phi) \times g r}}$$

where α = angle of inclination

$\tan\phi = \mu$

g = ~~coeffic~~ gravitational acceleration

r = radius of curve

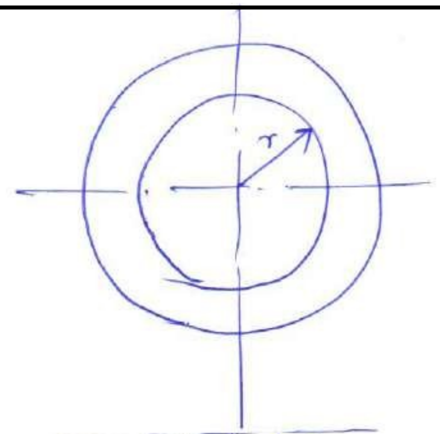
~~then the~~ vehicle will skid if the velocity is more than this value.

(b) condition for overturning:

Limiting speed from consideration of overturning

$$\boxed{v = \sqrt{g r \frac{G + (2he/G)}{2h - e}}}$$

Q.1 A circular ring has a mean radius r = 500 mm and is made of steel for which w = 77.12 kN/m³ and for which ultimate strength in tension is 413.85 MPa. Find the uniform speed of rotation about its geometrical axis perpendicular to the plane of the ring at which it will burst?

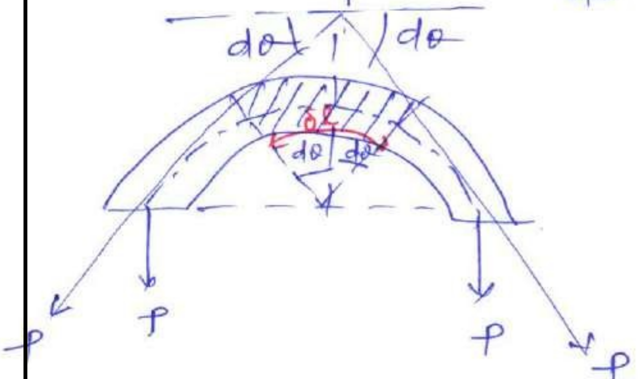


mean radius $r = 500\,mm = 0.5\,m$,

density of the wheel $w = 77.12\,kN/m^3$

σ_t = ultimate strength $= 413.85 \times 10^6\,Pa$

Now considering an infinitesimal small elementary ring extruded at an angle of $2d\theta$



centrifugal force acting

$$\delta F_c = \frac{dW}{g} \cdot \frac{v^2}{r}$$

Let P = tension on the ring

A = cross-sectional area of ring.

dW = wt. of the element

$= w \times volume$

$= w \times A \times \delta l$

$= w \times A \times r 2 d\theta$

Now centrifugal force

$$\frac{w}{g}(A\,dl) \times \frac{v^2}{r} = \frac{w}{g} \times A \cancel{r}\, 2d\theta \times \frac{v^2}{\cancel{r}} = \frac{2wA\,d\theta\, v^2}{g}$$

Balancing forces along the radius $= 2P \sin d\theta$

$$= \frac{2wA\,d\theta\,v^2}{g} \quad \text{---(1)}$$

as $d\theta$ is very small $\sin d\theta \simeq d\theta$

Eq. (1) may be written as

$$\cancel{2}P\,\cancel{d\theta} = \frac{\cancel{2}wA\,\cancel{d\theta} \cdot v^2}{g}$$

$$\Rightarrow \boxed{P = \frac{wAv^2}{g}} \quad \text{---(2)}$$

Tensile stress on the ring $\sigma_t = \frac{P}{A} = \frac{wv^2}{g}$

Now substituting the values

$$413.85 \times 10^6 = \frac{77.12 \times 10^3 \times v^2}{9.81} \Rightarrow v = 229.45\,m/s.$$

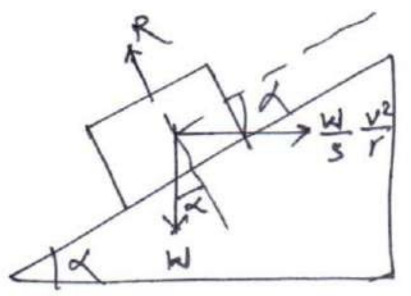
Now $v = \frac{\pi D N}{60} \Rightarrow N = \frac{60 \times 229.45}{\pi \times 1} = \boxed{4382.\ rpm}$

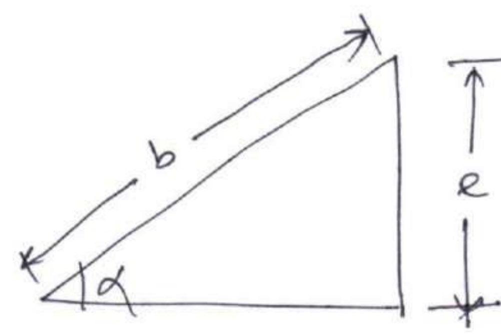
D' Alembert's Principle in Curvilinear Motion

Equation of motion of a particle may be written as

$$\left.\begin{aligned} X - m\ddot{x} &= 0 \\ Y - m\ddot{y} &= 0 \end{aligned}\right\} \quad \text{--- (1)}$$

Q.2 Find the proper super elevation 'e' for a 7.2 m highway curve of radius r = 600 m in order that a car travelling with a speed of 80 Kmph will have no tendency to skid sidewise.





$b = 7.2\,m \qquad r = 600\,m \qquad V = 80\,Kmph = 22.23\ m/s.$

Resolving along the inclined plane

$$W \sin\alpha = \frac{W}{g} \cdot \frac{v^2}{r} \cos\alpha$$

$$\Rightarrow \boxed{\tan\alpha = \frac{v^2}{rg}}$$

from the geometry $\sin\alpha = \frac{e}{b}$, since α is very small

let $\sin\alpha \simeq \tan\alpha$

$$\frac{v^2}{rg} = \frac{e}{b} \Rightarrow e = \frac{bv^2}{rg} = \frac{7.2 \times 22.23^2}{600 \times 9.81}$$

$$= 0.604\,m \quad \text{(Ans)}$$

Q.3 A racing car travels around a circular track of 300m radius with a speed of 384 kmph. What angle α should the floor of the track make with horizontal in order to safeguard against skidding.

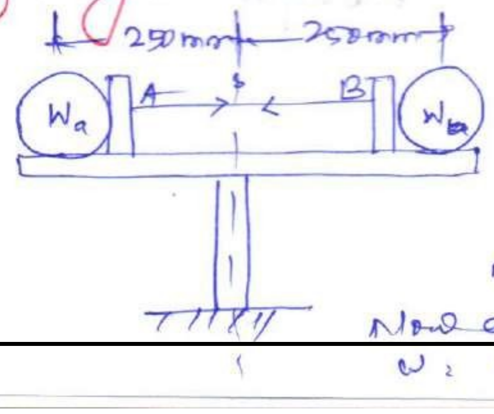
velocity $v = 384$ kmph $r = 300m$

$= 106.67$ m/s.

we have angle of banking $\tan\alpha = \frac{v^2}{rg}$

$\Rightarrow \alpha = \tan^{-1}\left(\frac{106.67^2}{300 \times 9.81}\right) = \boxed{75.5^\circ}$ (Ans)

Q.4 Two balls of wt $W_a = 44.5N$ and $W_b = 66.75N$ are connected by an elastic string and supported on a turntable as shown. When the turntable is at rest, the tension in the string is $S = 222.5N$ and the balls exert this same force on each of the stops A and B. What forces will they exert on the stops when the turntable is rotating uniformly about the vertical axis CD at 60 rpm ?



We have;

$W_a = 44.5N$ $W_b = 66.75N$

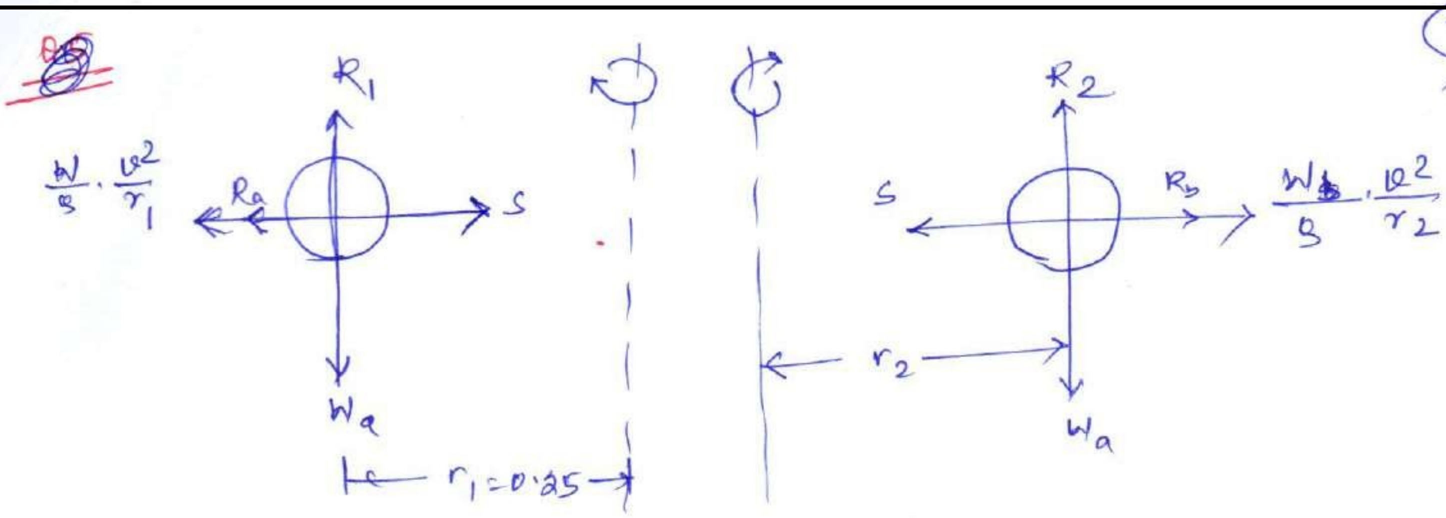
$S = 222.5N$

$N = 60$ rpm,

radius of rotation $r_1, r_2 = 0.25m$

Now angular velocity

$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 60}{60} = 2\pi$ rad/s.



~~Consider~~ Considering the left hand side ball

$$R_a + \frac{W_a}{g} \cdot r_1 w^2 = S$$

$$\Rightarrow R_a = 222.5 - \frac{44.5}{9.81} \times 0.25 \times (2\pi)^2$$

$$= \boxed{177.72\ N.}$$

Considering the ball on right hand side

$$R_b + \frac{W_b}{g} \times r_2 \times w^2 = S$$

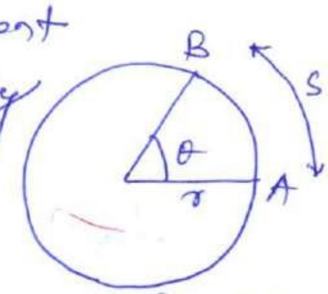
$$\Rightarrow R_b = 222.5 - \frac{66.75}{9.81} \times 0.25 \times (2\pi)^2$$

$$= \boxed{155.34\ N.}$$

Rotation of Rigid Bodies:-

Angular motion:-

The rate of change of angular displacement with time is called angular velocity and denoted by ω.



(fig-1)

$$\boxed{\omega = \frac{d\theta}{dt}} \quad —(1)$$

- The rate of change of angular velocity with time is called angular acceleration and denoted by α

$$\boxed{\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}} \quad —(2)$$

Angular acceleration may also be expressed as:

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \cdot \frac{d\theta}{dt}$$

$$\Rightarrow \boxed{\alpha = \omega \cdot \frac{d\omega}{d\theta}} \quad —(3) \quad \left(\because \frac{d\theta}{dt} = \omega\right)$$

Relationship between angular motion and linear motion

from fig-1 $\quad s = r\theta$

tangential velocity (linear) of the particle p.

$$\boxed{v = \frac{ds}{dt} = r \cdot \frac{d\theta}{dt}} \quad —(4)$$

linear acceleration $\boxed{a_t = \frac{dv}{dt} = r\frac{d^2\theta}{dt^2}} \quad —(5)$

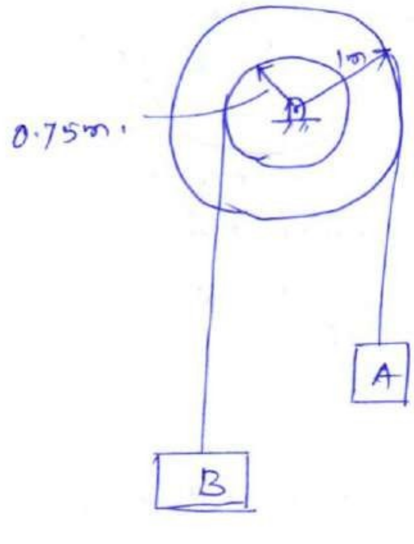
If $\frac{v^2}{r}$ = radial acceleration

Then $\boxed{a_n = \frac{v^2}{r} = r\omega^2}$ —(6) (where a_n = radial acceleration)

uniform angular velocity (ω)

$$\boxed{\omega = \frac{2\pi N}{60} \text{ rad/sec}} \quad —(7)$$

Q. The step pulley starts from rest and accelerates at 2 rad/s^2. How much time is required for block A to move 20m. Find also the velocity of A and B at that time.



When A moves by 20m, the angular displacement of pulley θ is given by

$$r\theta = s$$

$$\Rightarrow 1 \times \theta = 20$$

$$\Rightarrow \boxed{\theta = 20 \text{ rad}}$$

$\alpha = 2$ rad/s^2 and $\omega_0 = 0$

from kinematic relation

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\Rightarrow 20 = 0 \times t + \frac{1}{\cancel{2}} \times \cancel{2} t^2$$

$$\Rightarrow \boxed{t = 4.472 \text{ sec.}}$$

velocity of pulley at this time

$$\omega = \omega_0 + \alpha t$$

$$= 0 + 2 \times 4.472$$

$$= \boxed{8.944 \text{ rad/s}}$$

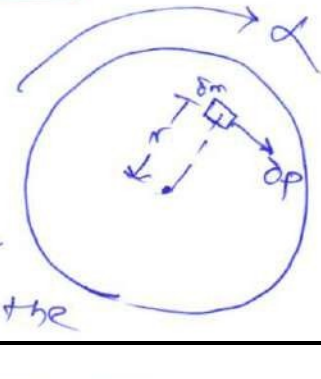
velocity of block A $v_A = 1 \times 8.944$

$$= \boxed{8.944 \text{ m/s}}$$

velocity of block B $v_B = 0.75 \times 8.944$

$$= \boxed{6.708 \text{ m/s.}}$$

Kinematics of rigid body for rotation:-



consider a wheel rotating about it's axis in clockwise direction with an acceleration α. Let δm be mass of an element at a distance r from the axis of rotation. δp be the resulting force on this element

$\delta p = \delta m \times a$ (a = tangential acceleration)

but $a = r \times \alpha$ (α = angular acceleration)

$\therefore \boxed{\delta p = \delta m \, r \, \alpha}$

Rotational moment $\delta M_t = \delta p \times r$

$= \delta m \, r^2 \alpha$

$M_t = \Sigma \, \delta M_t = \Sigma \, \delta m \, r^2 \alpha$

$= \alpha \, \Sigma \, \delta m \, r^2$

$= \alpha I$

$\Rightarrow \boxed{M_t = \alpha I}$ (I = mass moment of inertia)

Product of mass moment of inertia and angular velocity of rotating body is called angular momentum

so $\boxed{\text{Angular momentum} = I\omega}$

Kinetic energy of rotating bodies

$\boxed{K.E = \frac{1}{2} I \omega^2}$

Q.2 A flywheel weighing 50 kN and having radius of gyration 1 m losses its speed from 400 rpm to 280 rpm in 2 min. Calculate (a) retarding torque, (b) change in KE during the period, (c) change in angular momentum.

we have $\omega_0 = 400\,\text{rpm} = \frac{2\pi \times 400}{60} = 41.89$ rad/s

$\omega = 280\,\text{rpm} = \frac{2\pi \times 280}{60} = 29.32$ rad/s.

$t = 2\,\text{min} = 120$ sec,

$\omega = \omega_0 + \alpha t$

$\Rightarrow \alpha = \frac{\omega - \omega_0}{t} = \boxed{-.1047 \text{ rad/s}^2}$

Wt of flywheel = 50000 N

mass of " $= \frac{50000}{9.81} = 5096.84$ kg.

Radius of gyration $k = 1$ m.

$I = mk^2$

$= 5096.84 \times 1 = 5096.84$

(a) Retarding torque

$I\alpha = 5096.84 \times 0.1047$

$= \boxed{533.64 \text{ Nm.}}$

(b) change in KE

= initial KE - final KE

$= \frac{1}{2} I\omega_0^2 - \frac{1}{2} I\omega^2$

$= \frac{1}{2} \times 5096.84 \left(41.89^2 - 29.32^2\right)$

$= \boxed{\cancel{2280442.9} \text{ Nm}}$ $\boxed{2281115.462 \text{ Nm}}$

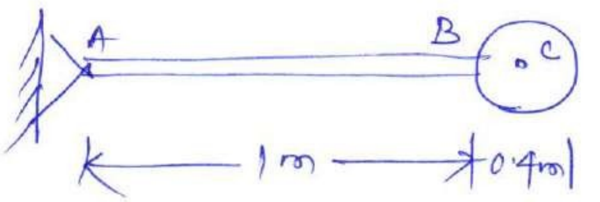
(c) change in angular momentum

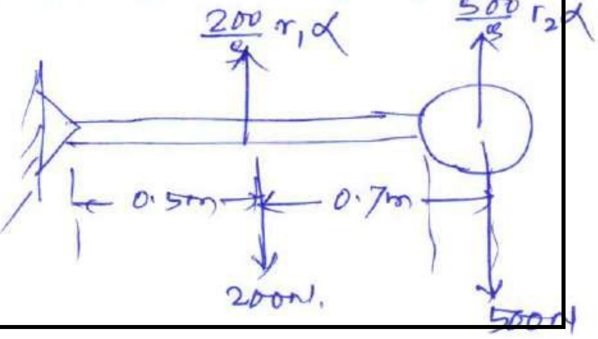
$I\omega_0 - I\omega$

$= 5096.84 \left(41.89 - 29.32\right)$

$= \boxed{64067.298 \text{ Nm.}}$

Q.3 A cylinder weighing 500N is welded to a 1 m long uniform bar of 200N. Determine the acceleration with which the assemby will rotate about point A; if released from rest in horizontal position. Determine the reactions at A at this instant.





Let α = angular acceleration of the assembly (3)

I = mass moment of inertia of the assembly

$I = I_G + Md^2$ (transfer formula)

mass MI of bar about A $= \frac{1}{2} \times \frac{200}{9.81} \times 1^2 + \frac{200}{9.81} \times (0.5)^2$

$= 6.7968$

mass MI of cylinder about A

$= \frac{1}{2} \frac{500}{9.81} \times 0.2^2 + \frac{500}{9.81} \times 1.2^2$

$= 74.4$

MI of the system $= 6.7968 + 74.4 = 81.2097$

Rotational moment about A

$M_t = 200 \times 0.5 + 500 \times 1.2 = 700$ N m.

$M_t = I\alpha$

$\Rightarrow \alpha = \frac{700}{81.2097} = \boxed{8.6197}$ rad/sec.

Instantaneous acceleration of rod AB is vertical and $= r_1 \alpha = 0.5 \times 8.6197$

$= 4.31$ m/s.

Similarly instantaneous acceleration of cylinder

$= r_2 \alpha = 1.2 \times 8.6197$

$= 10.34$ m/s.

Applying D'Alembert's dynamic equilibrium

$R_A = 200 + 500 - \frac{200}{9.81} \times 4.31 - \frac{500}{9.81} \times 10.34$

$\Rightarrow \boxed{R_A = 84.93 \text{ N.}}$ (Ans)

MODULE V

MECHANICAL VIBRATIONS

Definitions and Concepts

Amplitude : Maximum displacement from equilibrium position; the distance from the midpoint of a wave to its crest or trough.

Equilibrium position: The position about which an object in harmonic motion oscillates; the center of vibration.

Frequency: The number of vibrations per unit of time.

Hooke's law: Law that states that the restoring force applied by a spring is proportional to the displacement of the spring and opposite in direction.

Ideal spring: Any spring that obeys Hooke's law and does not dissipate energy within the spring.

Mechanical resonance: Condition in which natural oscillation frequency equals frequency of a driving force.

Period: The time for one complete cycle of oscillation.

Periodic motion: Motion that repeats itself at regular intervals of time.

Restoring force: The force acting on an oscillating object which is proportional to the displacement and always points toward the equilibrium position.

Simple harmonic motion: Regular, repeated, friction-free motion in which the restoring force has the mathematical form F = - kx.

Simple Harmonic Motion

A pendulum, a mass on a spring, and many other kinds of oscillators exhibit a special kind of oscillatory motion called Simple Harmonic Motion (SHM).

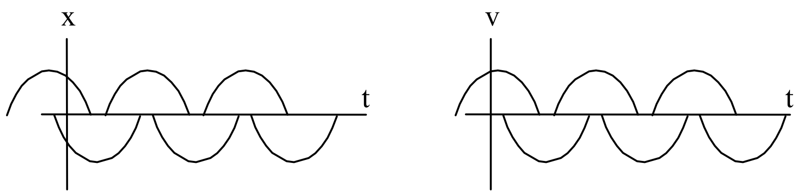
SHM occurs whenever :

i. here is a restoring force proportional to the displacement from equilibrium: $F \propto -x$

ii. the potential energy is proportional to the square of the displacement: $PE \propto x^2$

iii. the period T or frequency f = 1 / T is <u>independent</u> of the <u>amplitud</u>e of the motion.

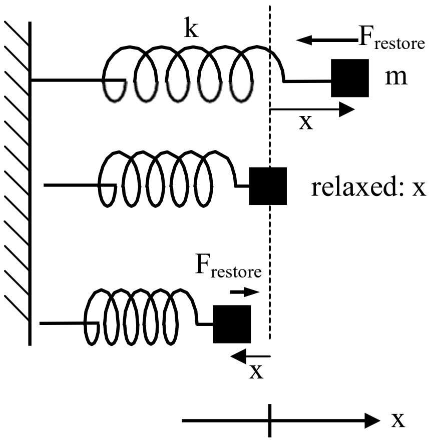
iv. the position *x*, the velocity *v*, and the acceleration *a* are all sinusoidal in time.



(*Sinusoidal* means sine, cosine, or anything in between.)
As we will see, any one of these four properties guarantees the other three. If one of these 4 things is true, then the oscillator is a simple harmonic oscillator and all 4 things must be true.

Not every kind of oscillation is SHM. For instance, a perfectly elastic ball bouncing up and down on a floor: the ball's position (height) is oscillating up and down, but none of the 4 conditions above is satisfied, so this is not an example of SHM.

A mass on a spring is the simplest kind of Simple Harmonic Oscillator.



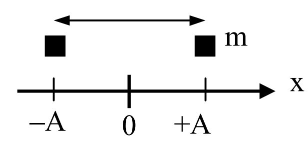
Hooke's Law: $\mathbf{F}_{spring} = -k\,\mathbf{x}$

(–) sign because direction of $\mathbf{F}_{spring}$ is opposite to the direction of displacement vector $\mathbf{x}$(**bold** font indicates vector)

k = spring constant = stiffness,
units [k] = N / m

Big k = stiff spring

Definition: *amplitude* A = $|x_{max}| = |x_{min}|$.



Mass oscillates between extreme positions x = +A and x = –A

Notice that Hooke's Law (F = –kx) is condition i : restoring force proportional to the displacement from equilibrium. We showed previously (Work and Energy Chapter) that for a spring obeying Hooke's Law, the potential energy is $U = (1/2)kx^2$, which is condition ii. Also, in the chapter on Conservation of Energy, we showed that F = –dU/dx, from which it follows that condition ii implies condition i. Thus, Hooke's Law and quadratic PE ($U \propto x^2$) are equivalent.

We now show that Hooke's Law guarantees conditions iii (period independent of amplitude) and iv (sinusoidal motion).

We begin by deriving the *differential equation* for SHM. A differential equation is simply an equation containing a derivative. Since the motion is 1D, we can drop the vector arrows and use sign to indicate direction.

$$F_{net} = ma \quad \text{and} \quad F_{net} = -kx \quad \Rightarrow \quad ma = -kx$$

$$a = dv/dt = d^2x/dt^2 \quad \Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

The constants k and m and both positive, so the k/m is always positive, always. For notational convenience, we write $k/m = \omega^2$. (The square on the ω reminds us that ω^2 is always positive.) The differential equation becomes

$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad \text{(equation of SHM)}$$

This is the *differential equation* for SHM. We seek a solution x = x(t) to this equation, a function x = x(t) whose second time derivative is the function x(t) multiplied by a negative constant ($-\omega^2 = -k/m$). The way you solve differential equations is the same way you solve integrals: you *guess* the solution and then check that the solution works.

Based on observation, sinusoidal solution: $x(t) = A\cos(\omega t + \varphi)$,

where A, φ are <u>any</u> constants and (as we'll show) $\omega = \sqrt{\frac{k}{m}}$.

A = amplitude: x oscillates between +A and –A

φ = phase constant (more on this later)

Danger: ωt and φ have units of radians (not degrees). So set your calculators to radians when using this formula.

Just as with circular motion, the angular frequency ω for SHM is related to the period by

$\omega = 2\pi f = \frac{2\pi}{T}$, T = period.

(What does SHM have to do with circular motion? We'll see later.)

Let's check that $x(t) = A\cos(\omega t + \varphi)$ is a solution of the SHM equation.

Taking the first derivative dx/dt , we get $v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \varphi)$.

Here, we've used the Chain Rule: $\frac{d}{dt}\cos(\omega t + \varphi) = \frac{d\cos(\theta)}{d\theta}\frac{d\theta}{dt}, \quad (\theta = \omega t + \varphi)$

$$= -\sin\theta\cdot\omega = -\omega\sin(\omega t + \varphi)$$

Taking a second derivative, we get

$$a(t) = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{d}{dt}\left(-A\omega\sin(\omega t + \varphi)\right) = -A\omega^2\cos(\omega t + \varphi)$$

$$\frac{d^2x}{dt^2} = -\omega^2\left[A\cos(\omega t + \varphi)\right]$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

This is the SHM equation, with $\omega^2 = \frac{k}{m}$, $\omega = \sqrt{\frac{k}{m}}$

We have shown that our assumed solution is indeed a solution of the SHM equation. (I leave to the mathematicians to show that this solution is unique. Physicists seldom worry about that kind of thing, since we know that nature usually provides only one solution for physical systems, such as masses on springs.)

We have also shown condition iv: x, v, and a are all sinusoidal functions of time:

$$x(t) = A\cos(\omega t + \varphi)$$
$$v(t) = -A\omega\sin(\omega t + \varphi)$$
$$a(t) = -A\omega^2\cos(\omega t + \varphi)$$

The period T is given by $\omega = \sqrt{\dfrac{k}{m}} = \dfrac{2\pi}{T} \Rightarrow T = 2\pi\sqrt{\dfrac{m}{k}}$. We see that T does not depend on the amplitude A (condition iii).

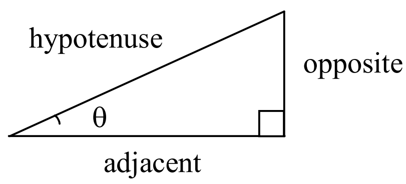
Let's first try to make sense of $\omega = \sqrt{k/m}$: big ω means small T which means rapid oscillations. According to the formula, we get a big ω when k is big and m is small. This makes sense: a big k (stiff spring) and a small mass m will indeed produce very rapid oscillations and a big ω.

A closer look atx(t) = A cos(ωt+φ)

Let's review the sine and cosine functions and their relation to the *unit* circle. We often define the sine and cosine functions this way:

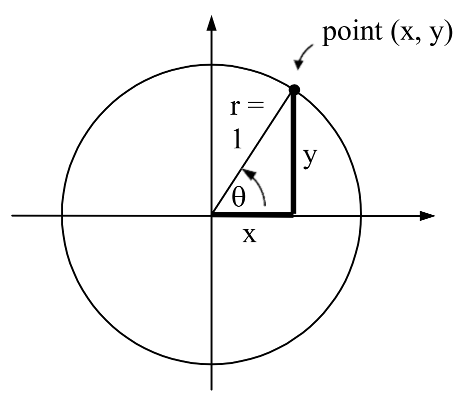
$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$



This way of defining sine and cosine is correct but incomplete. It is hard to see from this definition how to get the sine or cosine of an angle greater than 90°.

A more complete way of defining sine and cosine, a way that gives the value of the sine and cosine for *any* angle, is this: Draw a *unit* circle (a circle of radius r = 1) centered on the origin of the x-y axes as shown:

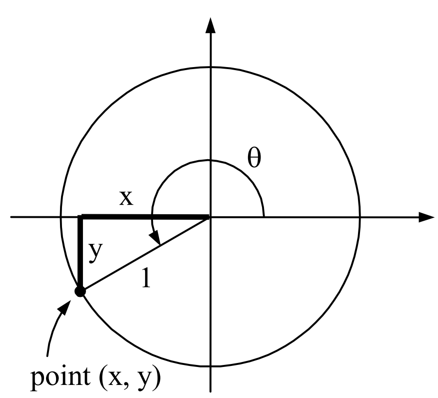


Define sine and cosine as

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x$$
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y$$

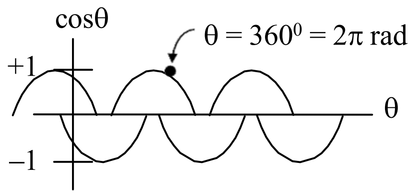
This way of defining sin and cos allows us to compute the sin or cos of *any* angle at all.

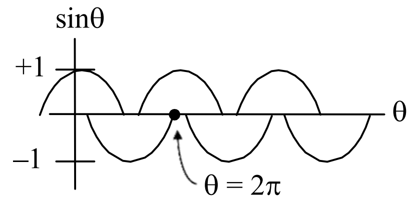
For instance, suppose the angle is θ = 210°. Then the diagram looks like this:



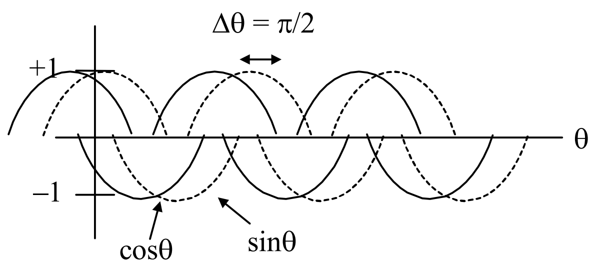
The point on the unit circle is in the third quadrant, where both x and y are negative. So both cosθ = x and sinθ = y are negative

For any angle θ, even angles bigger than 360° (more than once around the circle), we can always compute sin and cos. When we plot sin and cos vs angle θ, we get functions that oscillate between +1 and –1 like so:





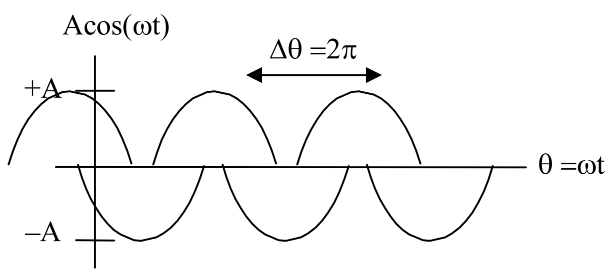
We will almost always measure angle θ in radians. Once around the circle is 2π radians, so sine and cosine functions are periodic and repeat every time θ increases by 2π rad. The sine and cosine functions have exactly the same shape, except that sin is shifted to the right compared to cos by $\Delta\theta = \pi/2$. Both these functions are called *sinusoidal* functions.



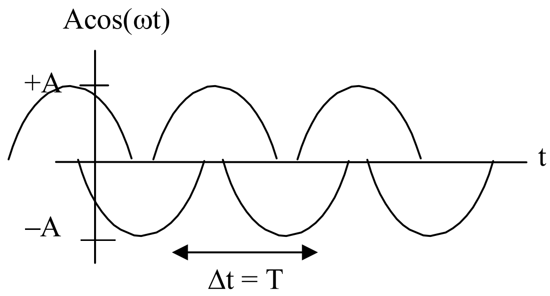
The function $\cos(\theta + \varphi)$ can be made to be anything in between $\cos(\theta)$ and $\sin(\theta)$ by adjusting the size of the *phase*φ between 0 and −2π.

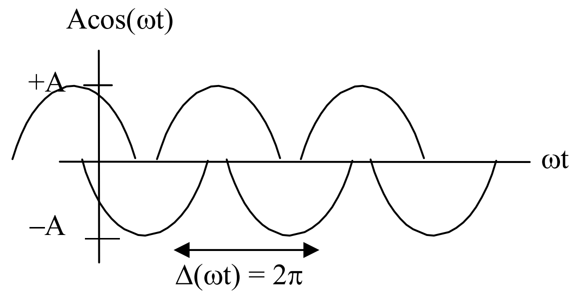
$$\cos\theta,\ (\varphi = 0)\ \rightarrow\ \sin\theta = \cos\left(\theta - \frac{\pi}{2}\right),\quad (\varphi = -\pi/2)$$

The function $\cos(\omega t + \varphi)$ oscillates between +1 and −1, so the function $A\cos(\omega t + \varphi)$ oscillates between +A and −A.



Why $\omega = \dfrac{2\pi}{T}$? The function $f(\theta) = \cos\theta$ is periodic with period $\Delta\theta = 2\pi$. Since $\theta = \omega t + \varphi$, and φ is some constant, we have $\Delta\theta = \omega\,\Delta t$. One complete cycle of the cosine function corresponds to $\Delta\theta = 2\pi$ and $\Delta t = T$, (T is the period). So we have $2\pi = \omega T$ or $\omega = \dfrac{2\pi}{T}$. Here is another way to see it: $\cos(\omega t) = \cos\left(2\pi \dfrac{t}{T}\right)$ is periodic with period $\Delta t = T$. To see this, notice that when t increases by T, the fraction t/T increases by 1 and the fraction 2πt/T increases by 2π.





Now back to simple harmonic motion. Instead of a circle of radius 1, we have a circle of radius A (where A is the amplitude of the Simple Harmonic Motion).

SHM and Conservation of Energy:

Recall $PE_{elastic} = (1/2)\,k\,x^2$ = work done to compress or stretch a spring by distance x.

If there is no friction, then the total energy $E_{tot} = KE + PE$ = constant during oscillation. The value of E_{tot} depends on initial conditions – where the mass is and how fast it is moving initially. But once the mass is set in motion, E_{tot} stays constant (assuming no dissipation.)

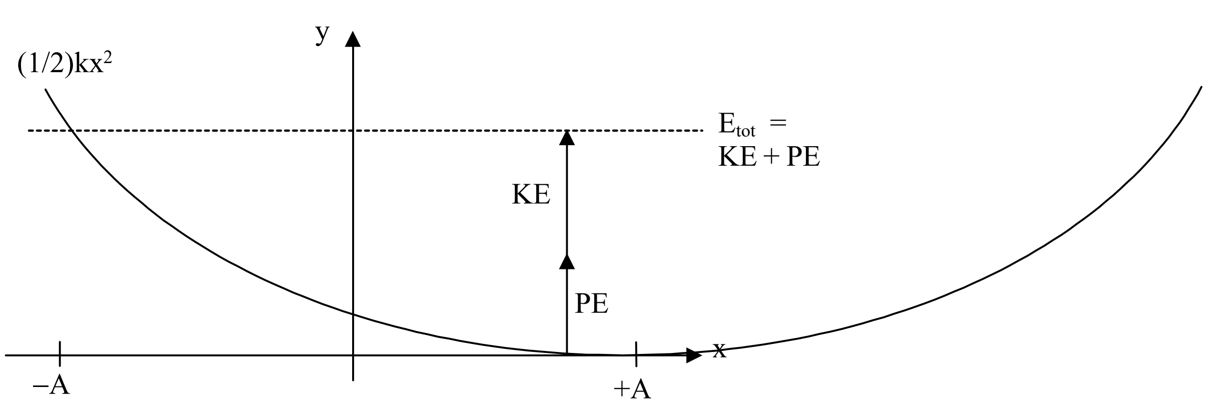
At any position x, speed v is such that $\boxed{\tfrac{1}{2}\,m\,v^2 + \tfrac{1}{2}\,k\,x^2} = E_{tot}$.

When |x| = A, then v = 0, and all the energy is PE: $\underbrace{KE}_{0} + \underbrace{PE}_{(1/2)kA^2} = E_{tot}$

So total energy $E_{tot} = \tfrac{1}{2}\,k\,A^2$

When x = 0, v = v_{max}, and all the energy is KE: $\underbrace{KE}_{(1/2)mv_{max}^2} + \underbrace{PE}_{0} = E_{tot}$

So, total energy $E_{tot} = \tfrac{1}{2}\,m\,v_{max}^2$.



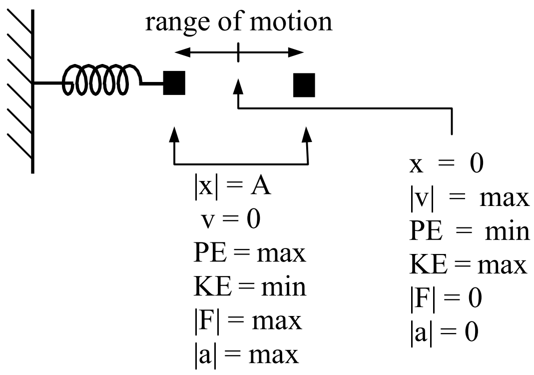
So, we can relate v_{max} to amplitude A : $PE_{max} = KE_{max} = E_{tot} \Rightarrow \tfrac{1}{2}\,k\,A^2 = \tfrac{1}{2}\,m\,v_{max}^2 \Rightarrow$

$$v_{max} = \sqrt{\frac{k}{m}}\;A$$

Example Problem: A mass m on a spring with spring constant k is oscillating with amplitude A. Derive a general formula for the speed v of the mass when its position is x.

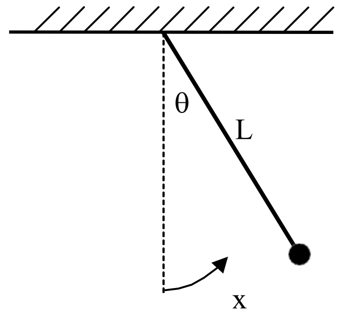
Answer: $v(x) = A\sqrt{\dfrac{k}{m}}\sqrt{1-\left(\dfrac{x}{A}\right)^2}$

Be sure you understand these things:



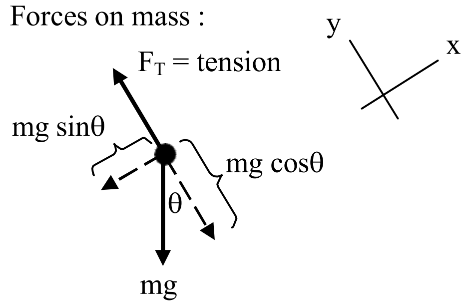
Pendulum Motion

A simple pendulum consists of a small mass m suspended at the end of a massless string of length L. A pendulum executes SHM, <u>if</u>the amplitude is not too large.



$\theta = x / L$ (rads)

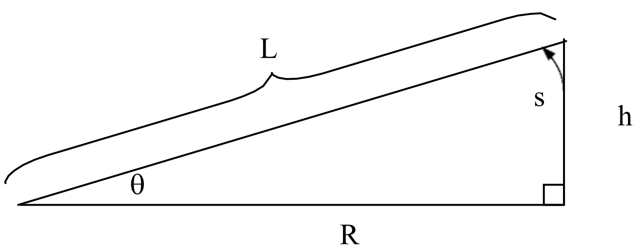
Forces on mass :



The restoring force is the component of the force along the direction of motion:

$$\text{restoring force} = -mg\sin\theta \cong -mg\,\theta = -mg\frac{x}{L}$$

Claim: $\sin\theta \cong \theta$ (rads) when θ is small. $\sin\theta = \dfrac{h}{L}$



If θ small, then $h \approx s$, and $L \approx R$,
so $\sin\theta \approx \theta$.

Try it on your calculator: $\theta = 5^\circ = 0.087266..$rad, $\sin\theta = 0.087156..$

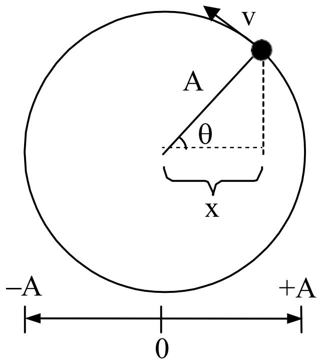
$F_{restore} = -\left(\dfrac{mg}{L}\right)x$ is exactly like Hooke's Law $F_{restore} = -k\,x$, except we have replaced the constant k with another constant (mg / L). The math is exactly the same as with a mass on a spring; all results are the same, except we replace k with (mg/L).

$$T_{spring} = 2\pi\sqrt{\frac{m}{k}} \;\Rightarrow\; T_{pend} = 2\pi\sqrt{\frac{m}{(mg/L)}} = 2\pi\sqrt{\frac{L}{g}}$$

Notice that the period is independent of the amplitude; the period depends only on length L and acceleration of gravity. (But this is true only if θ is not too large.)

SHM and circular motion

There is an exact analogy between SHM and *circular motion*. Consider a particle moving with constant speed v around the rim of a circle of radius A.
The x-component of the position of the particle has *exactly* the same mathematical form as the motion of a mass on a spring executing SHM with amplitude A.



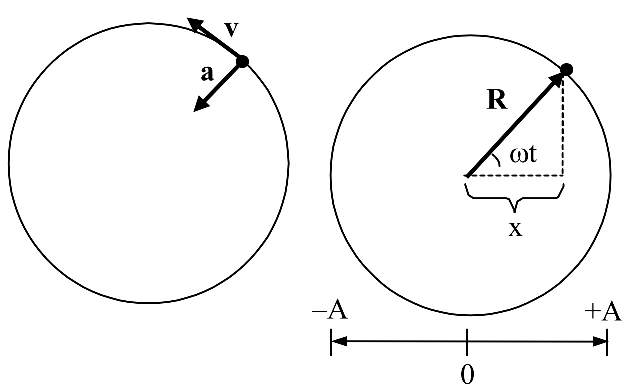
Angular velocity $\omega = \dfrac{d\theta}{dt} = \text{const}$

$\theta = \omega t$ so

This same formula also describes the *sinusoidal* motion of a mass on a spring.

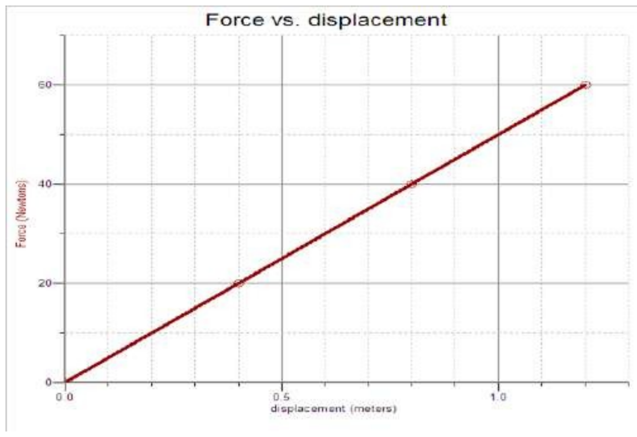
That the same formula applies for two different situations (mass on a spring & circular motion) is no accident. The two situations have the same solution because they both obey the same equation. As Feynman said, "The same equations have the same solutions". The equation of SHM is $\dfrac{d^2x}{dt^2} = -\omega^2 x$. We now show that a particle in circular motion obeys this same SHM equation.

Recall that for circular motion with angular speed ω, the acceleration of a the particle is toward the center and has magnitude $|\vec{a}| = \dfrac{v^2}{R}$. Since $v = \omega R$, we can rewrite this as $|\vec{a}| = \dfrac{(\omega R)^2}{R} = \omega^2 R$



Let's set the origin at the center of the circle so the position vector **R** is along the radius. Notice that the acceleration vector **a** is always in the direction opposite the position vector **R**. Since $|\vec{a}| = \omega^2 |\vec{R}|$, the vectors **a** and **R** are related by $\vec{a} = -\omega^2 \vec{R}$. The x-component of this vector equation is: $a_x = -\omega^2 R_x$. If we write $R_x = x$, then we have $\dfrac{d^2x}{dt^2} = -\omega^2 x$, which is the SHM equation. Done.

Example
A mass of 0.5 kg oscillates on the end of a spring on a horizontal surface with negligible friction according to the equation $x = A\cos(\omega t)$. The graph of *F vs. x* for this motion is shown below.



The last data point corresponds to the maximum displacement of the mass.
Determine the
(a) angular frequency ω of the oscillation,
(b) frequency f of oscillation,
(c) amplitude of oscillation,
(d) displacement from equilibrium position ($x = 0$) at a time of 2 s.

Solution:
(a) We know that the spring constant $k = 50$ N/m from when we looked at this graph earlier. So,

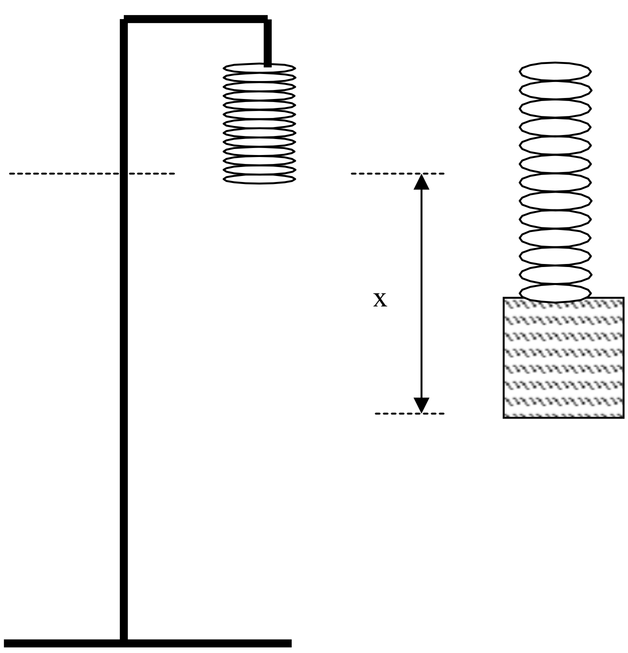
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{50\,N/m}{0.5\,kg}} = 10\,\frac{rad}{s}$$

(b) $f = \dfrac{\omega}{2\pi} = \dfrac{10\,rad/s}{2\pi} = 1.6\,Hz$

(c) The amplitude corresponds to the last displacement on the graph, $A = 1.2$ m.

(d) $x = A\cos(\omega t) = (1.2\,m)\cos[(10\,rad/s)(2\,s)] = 0.5\,m$

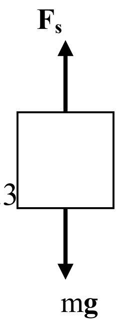
Example
A spring of constant $k = 100$ N/m hangs at its natural length from a fixed stand. A mass of 3 kg is hung on the end of the spring, and slowly let down until the spring and mass hang at their new equilibrium position.



(a) Find the value of the quantity x in the figure above.The spring is now pulled down an additional distance x and released from rest.
(b) What is the potential energy in the spring at this distance?
(c) What is the speed of the mass as it passes the equilibrium position?
(d) How high above the point of release will the mass rise?
(e) What is the period of oscillation for the mass?

Solution:
(a) As it hangs in equilibrium, the upward spring force must be equal and opposite to the downward weight of the block.

$$F_s = mg$$
$$kx = mg$$
$$x = \frac{mg}{k} = \frac{(3kg)(10m/s^2)}{100\ N/m} = 0.3$$



(b) The potential energy in the spring is related to the displacement from equilibrium position by the equation
$$U = \frac{1}{2}kx^2 = \frac{1}{2}(100\,N/m)(0.3m)^2 = 4.5\,J$$
(c) Since energy is conserved during the oscillation of the mass, the kinetic energy of the mass as it passes through the equilibrium position is equal to the potential energy at the amplitude. Thus,

$$K = U = \frac{1}{2}mv^2$$
$$v = \sqrt{\frac{2U}{m}} = \sqrt{\frac{2(4.5J)}{3kg}} = 1.7\,m/s$$
(d) Since the amplitude of the oscillation is 0.3 m, it will rise to 0.3 m above the equilibrium position.
$$\text{(e) } T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{3\,kg}{100\,N/m}} = 1.1s$$

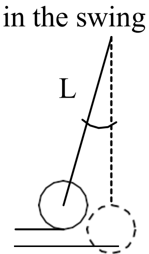
Example
A pendulum of mass 0.4 kg and length 0.6 m is pulled back and released from and angle of 10˚ to the vertical.

(a) What is the potential energy of the mass at the instant it is released. Choose potential energy to be zero at the bottom of the swing.
(b) What is the speed of the mass as it passes its lowest point?

This same pendulum is taken to another planet where its period is 1.0 second.
(c) What is the acceleration due to gravity on this planet?

Solution
(a) First we must find the height above the lowest point in the swing at the instant the pendulum is released.

Recall from chapter 1 of this study guide that $h = L - L\cos\theta$.
Then
$$U = mg(L - L\cos\theta)$$
$$U = (0.4kg)(10m/s^2)(0.6m - 0.6m\cos10°) = 0.$$



(b) Conservation of energy:

$$U_{max} = K_{max} = \frac{1}{2}mv^2$$

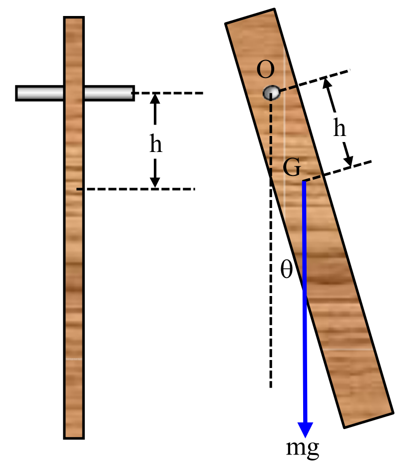
$$v = \sqrt{\frac{2U}{m}} = \sqrt{\frac{2(0.4\,J)}{0.4\,kg}} = 1.4\,m/s$$

(c)

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

$$g = \frac{4\pi^2 L}{T^2} = \frac{4\pi^2(0.6\,m)}{(1.0\,s)^2} = 23.7\,\frac{m}{s^2}$$

COMPOUND PENDULUM



AIM:

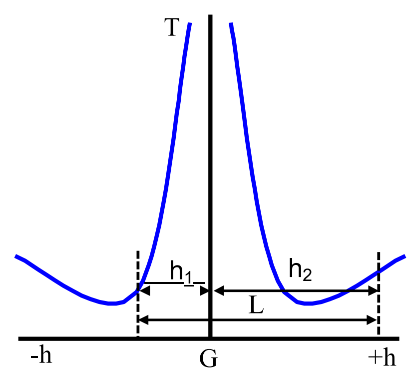
The aim of this experiment is to measure g using a compound pendulum.

YOU WILL NEED:

WHAT TO DO:

First put the knife edge through the hole in the metre rule nearest end A, and record the time for 10 oscillations. Hence work out the time for one oscillation (T).

Repeat this for each hole in the ruler for a series of different distances (d) from end A.



ANALYSIS AND CALCULATIONS:

Plot a graph of T against d.

From the graph record a series of values of the simple equivalent pendulum (L).

Calculate the value of g from the graph or from the formula:

$$T^2 = 4\pi^2 L/g$$

Torsion Pendulum:

1. Introduction

Torsion is a type of stress, which is easier to explain for a uniform wire or a rod when one end of the wire is fixed, and the other end is twisted about the axis of the wire by an external force. The external force causes deformation of the wire and appearance of counterforce in the material. If this end is released, the internal torsion force acts to restore the initial shape and size of the wire. This behavior is similar to the one of the released end of a linear spring with a mass attached.

Attaching a mass to the twisting end of the wire, one can produce a torsion pendulum with circular oscillation of the mass in the plane perpendicular to the axis of the wire.

To derive equations of rotational motion of the torsion pendulum, it would be useful to recall a resemblance of quantities in linear and rotational motion. We know that if initially a mass is motionless, its linear motion is caused by force F; correspondingly, if an extended body does not rotate initially, its rotation is caused by torque τ. The measure of inertia in linear motion is mass, m, while the measure of inertia in rotational motion is the moment of inertia about an axis of rotation, I. For linear and angular displacement in a one-dimensional problem, we use either x or θ. Thus, the two equations of motion are:

$$F_x = ma_x \text{and} \quad \tau = I\alpha \qquad (1)$$

where a_x and α are the linear and the angular acceleration.
If the linear motion is caused by elastic, or spring, force, the Hooke's law gives $F_x = -kx$, where k is the spring constant. If the rotation is caused by torsion, the Hooke's law must result in

$\tau = -\kappa\theta$

where κ is the torsion constant, or torsional stiffness, that depends on properties of the wire. It is essentially a measure of the amount of torque required to rotate the free end of the wire 1 radian.

Your answer to the Preparatory Question 2 gives the following relationship between the moment of inertia I of an oscillating object and the period of oscillation T as:

This relationship is true for oscillation where damping is negligible and can be ignored. Otherwise the relationship between I and κ is given by

$$I = \frac{\kappa}{\omega_0^2} \qquad (3^*)$$

where ω_0 can be found from $\omega = \sqrt{\omega_0^2 - \left(\frac{c}{2I}\right)^2}$ (3**)

$\omega = \frac{2\pi}{T} = 2\pi f$; f is the frequency of damped oscillation; and c is the *damping coefficient*.
The relationship between the torsion constant κ and the diameter of the wire d is given in [3] (check your answer to the Preparatory Question 1) as

$$\kappa = \frac{\pi G d^4}{32 l} \qquad (4)$$

where l is the length of the wire and G is the shear modulus for the material of the wire.

As any mechanical motion, the torsional oscillation is damped by resistive force originating from excitation of thermal modes of oscillation of atoms inside the crystal lattice of the wire and air resistance to the motion of the oscillating object. We can estimate the torque of the resistive force as being directly proportional to the angular speed of the twisting wire, i.e. the torque $\tau_R = -c\,d\theta/dt$ (recall the drag force on mass on spring in viscose medium as $R = -bv$). Combining Eq.(1), (2) and the expression for τ_R, we obtain the equation of motion of a torsional pendulum as follows:

$$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + \kappa\theta = 0 \qquad (5)$$

The solution of Eq.(5) is similar to the solution of the equation for damped oscillation of a mass on spring and is given by:

$$\theta = Ae^{-\alpha t}\cos(\omega t + \phi) \qquad (6)$$

where $\alpha = c/2I$ (7)

and $\alpha = \beta^{-1}$ with β being the time constant of the damped oscillation; c is the damping coefficient; ω is the angular frequency of torsional oscillation measured in the experiment; and φ can be made zero by releasing the object on the wire at a position of the greatest deviation from equilibrium.

Equation (6) can be used to calculate c (damping coefficient) and β (time constant = amount of time to decay e times) with DataStudio interface and software.

Another important formula is $\alpha = \omega_0/2Q$, where Q is the *quality factor* and $\omega_0^2 = \kappa / I$ (see Eq.3'). The ratio

$$\zeta = \alpha/\omega_0 = (2Q)^{-1} \qquad (8)$$

is called the *damping ratio*.

Free vibration of One Degree of Freedom Systems

Free vibration of a system is vibration due to its own ***internal forces*** (free of external impressive forces). It is initiated by an initial deviation (an energy input) of the system from its static equilibrium position. Once the initial deviation (a displacement or a velocity or both) is suddenly withdrawn, the strain energy stored in the system forces the system to return to its original, static equilibrium configuration. Due to the inertia of the system, the system will not return to the equilibrium configuration in a straightforward way. Instead it will oscillate about this position — free vibration.

A system experiencing free vibration oscillates at one or more of its natural frequencies, which are properties of its mass and stiffness distribution. If there is no damping (an undamped system), the system vibrates at the (*undamped*) frequency (frequencies) forever. Otherwise, it vibrates at the (*damped*) frequency (frequencies) and dies out gradually. When damping is not large, as in most cases in engineering, undamped and damped frequencies are very close. Therefore usually no distinction is made between the two types of frequencies.

The number of natural frequencies of a system equals to the number of its degrees-of-freedom. Normally, the low frequencies are more important.

Damping always exists in materials. This damping is called material damping, which is always positive (dissipating energy). However, air flow, friction and others may 'present' negative damping.

Undamped Free Vibration

Equation of motion based on the free-body diagram

$$m\ddot{x} + kx = 0$$

$$\downarrow$$

$$\ddot{x} + \omega_n^2 x = 0 \qquad \omega_n = \sqrt{\frac{k}{m}}$$

natural frequency

$$T = 2\pi\sqrt{\frac{m}{k}}$$

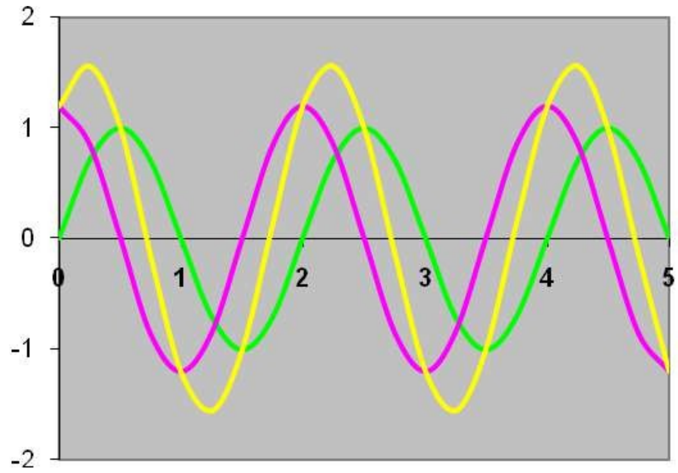
period

$$x(t) = A\sin\omega_n t + B\cos\omega_n t$$

A and B are determined by the *initial conditions.*

Sin or Cos

$$\boxed{\begin{array}{ll} l = ? & \omega_n = ? \\ x(0) = ? & \dot{x}(0) = ? \end{array}}$$



$$x(t) = \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t + x(0)\cos\omega_n t$$

$$= \sqrt{\left(\frac{\dot{x}(0)}{\omega_n}\right)^2 + [x(0)]^2}\,\sin(\omega_n t + \phi) \quad \text{where} \quad \phi = \arctan\left(\frac{x(0)\omega_n}{\dot{x}(0)}\right)$$

Vibration of a pendulum

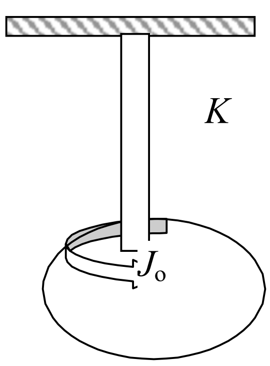
How to establish the equation of motion?

What is its natural frequency?

$$ml^2\ddot{\theta} = -mgl\sin\theta \rightarrow \ddot{\theta} + g\sin\theta = 0$$

$$\ddot{\theta} + g\theta = 0 \quad \rightarrow \quad \omega_n = \sqrt{\frac{g}{l}}$$

Systems with Rotational Degrees-of-Freedom



Equation of Motion

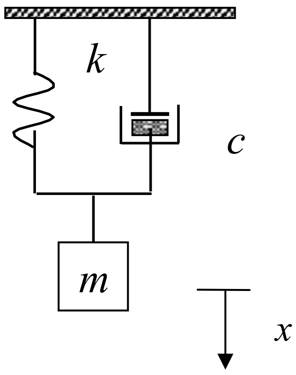
$$J_o\ddot{\theta} + K\theta = 0$$

natural frequency

$$\omega_n = \sqrt{\frac{K}{J_o}}$$

Systems involving rotational degrees-of-freedom are always more difficult to deal with, in particular when translational degrees-of-freedom are also present. Gear care is needed to identify both degrees-of-freedom and construct suitable equations of motion.

Damped Free Vibration (first hurdle in studying vibration)



$$m\ddot{x} = -kx - c\dot{x} \qquad m\ddot{x} + c\dot{x} + kx = 0$$

standard equation $$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

damping factor $$\zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}}$$

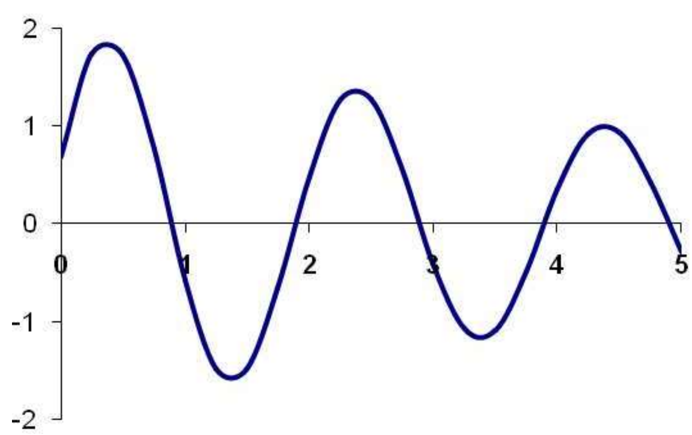
1. **oscillatory motion** (under-damped $\zeta < 1$)

$$x(t) = \exp(-\zeta\omega_n t)[C_1 \exp(\sqrt{\zeta^2 - 1}\,\omega_n t) + C_2 \exp(-\sqrt{\zeta^2 - 1}\,\omega_n t)]$$

$$x(t) = \exp(-\zeta\omega_n t)(A\sin\omega_d t + B\cos\omega_d t) = X\exp(-\zeta\omega_n t)\sin(\omega_d t + \phi)$$

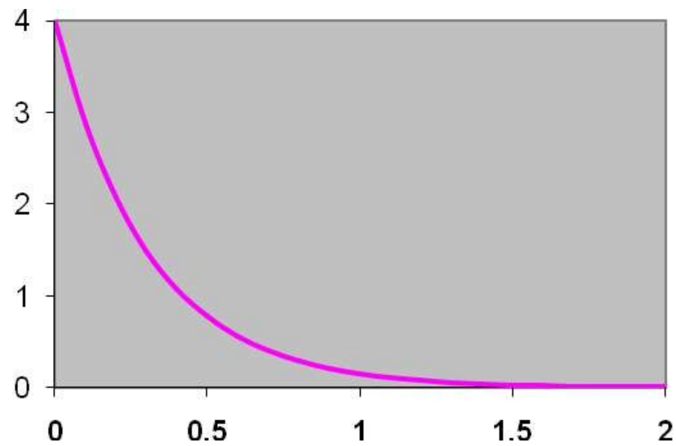
$$x(t) = \exp(-\zeta\omega_n t)\left[\frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}\sin\omega_d t + x(0)\cos\omega_d t\right] \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

damped natural frequency

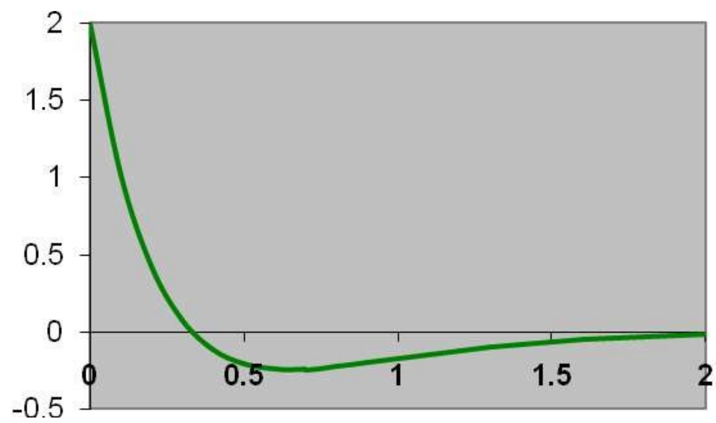


2. **nonoscillatory motion** (over-damped $\zeta > 1$)

$$x(t) = \exp(-\zeta\omega_n t)[A\exp(\sqrt{\zeta^2 - 1}\,\omega_n t) + B\exp(-\sqrt{\zeta^2 - 1}\,\omega_n t)]$$



3. **critically damped motion** ($\zeta = 1$)

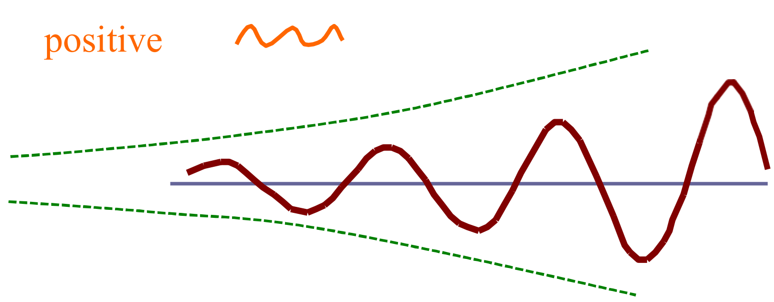


$$x(t) = (A + Bt)\exp(-\omega_n t)$$

4. negative damping of $\zeta < 0$ as a special case of $\zeta < 1$:

$$x(t) = \exp(-\zeta\omega_n t)[C_1 \exp(\sqrt{\zeta^2 - 1}\,\omega_n t) + C_2 \exp(-\sqrt{\zeta^2 - 1}\,\omega_n t)]$$

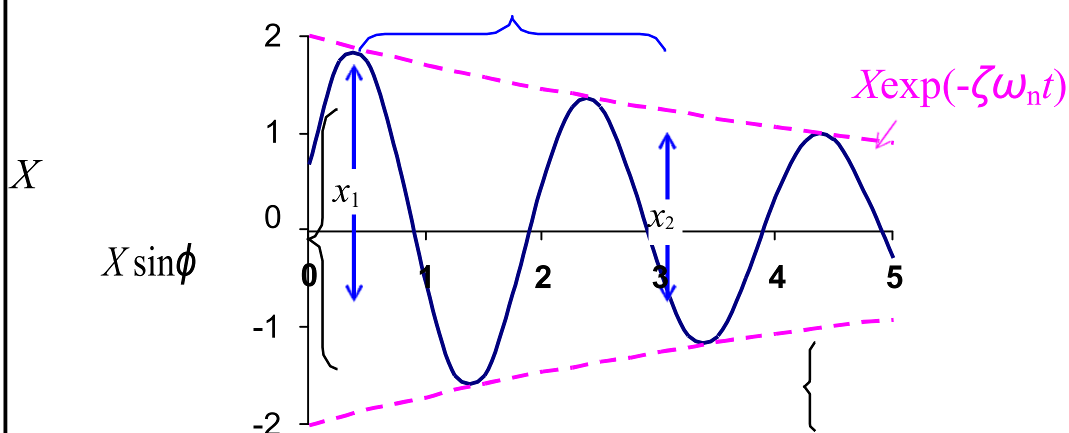
positive



Divergent oscillatory motion (flutter) due to negative damping

Determination of Damping

$$x(t) = X\exp(-\zeta\omega_n t)\sin(\omega_d t + \phi)$$



$$2\exp(-0.05\pi t)\sin(0.9988\,\pi t + \phi)$$

two consecutive peaks:

$$x_1 = X\exp(-\zeta\omega_n t_1)\sin(\omega_d t_1 + \phi)$$

$$x_2 = X\exp(-\zeta\omega_n t_2)\sin(\omega_d t_2 + \phi) = X\exp(-\zeta\omega_n t_2)\sin(\omega_d t_1 + \phi)$$

logarithm decrement$\Rightarrow$ $\delta = \ln\dfrac{x_1}{x_2} = \zeta\omega_n\tau_d \quad \zeta = \dfrac{\delta}{\omega_n\tau_d}$

Example:
The 2nd and 4th peaks of a damped free vibration measured are respectively 0.021 and 0.013. What is damping factor?

Solution:

$$\frac{x(t_2)}{x(t_4)} = \exp(\zeta\omega_n 2\tau_d) \quad \rightarrow \quad 2\zeta\omega_n\tau_d = \ln\left(\frac{x(t_2)}{x(t_4)}\right)$$

$$2\zeta\omega_n\tau_d = 2\zeta\omega_n\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{4\pi\zeta}{\sqrt{1-\zeta^2}} = \ln\left(\frac{x(t_2)}{x(t_4)}\right)$$

If a small damping is assumed, $2\zeta\omega_n\tau_d = 4\pi\zeta = \ln\left(\dfrac{x(t_2)}{x(t_4)}\right)$. This leads to

$$\zeta = \frac{1}{4\pi}\ln\left(\frac{x(t_2)}{x(t_4)}\right) = 0.0382 = 3.82\%.$$

If such an assumption is not made, then $\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{1}{4\pi}\ln\left(\frac{x(t_2)}{x(t_4)}\right)$ and hence $\frac{\zeta^2}{1-\zeta^2} = \left[\frac{1}{4\pi}\ln\left(\frac{x(t_2)}{x(t_4)}\right)\right]^2$. This leads to

$$\zeta = \frac{\frac{1}{4\pi}\ln\left(\frac{x(t_2)}{x(t_4)}\right)}{\sqrt{1+\left[\frac{1}{4\pi}\ln\left(\frac{x(t_2)}{x(t_4)}\right)\right]^2}} = 0.0381 = 3.81\%$$

. So virtually the same value.

General differential equations

$$a_n\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \ldots\ldots + a_1\frac{dx}{dt^1} + a_0 = 0$$

first solve the characteristic equation

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots\ldots + a_1\lambda + a_0 = 0$$

If all roots λ_j are distinct, then the general solution is

$$x(t) = \sum_{j=1}^{n} b_j \exp(\lambda_j t)$$

where b_j are constants to be determined.

If there are repeated roots, t^m (integer $m > 1$) appears in a solution. These are not interesting cases for mechanical vibration.

λ in response to the change of a parameter reveal stability properties